RATIONAL SOLUTIONS TO A SYSTEM OF COUPLED PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Two types of high-order rational solutions to a system of coupled partial differential equations, namely general high-order rogue wave solutions and $W$-shaped soliton solutions, are derived via the Hirota bilinear method. These rational solutions are given in terms of determinants whose matrix elements have plain algebraic expressions. It is shown that the general $N$-th order rogue waves contain $N - 1$ free irreducible complex parameters. The $N$-th order $W$-shaped rational solitons consist of $N$ parallel $W$-shaped line waves.

Key words: coupled systems, rogue waves, $W$-shaped solitons, bilinear method.

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1. INTRODUCTION

The study of nonlinear evolution equations (NLEEs) has made enormous advances in the areas of engineering and physical and mathematical sciences [1–3]. These equations appear in fluid dynamics, plasma physics, nonlinear optics, nuclear physics, mathematical biosciences, just to name a few. Therefore, the investigation of various solutions to NLEEs has been one of main problems during the past years, including rational solution [4–10], algebro-geometric solution [11] and so on. Indeed, various effective methods have been developed to derive exact solutions to NLEEs, such as the Darboux transformation method [12, 13], the inverse scattering method [14], the Hirota bilinear method [15], the homogeneous balance method [16, 17], the Lie group method [18, 19] and so on [20–22].

The study of nonlinear waves and soliton theory has become a more and more significant subject in many branches of nonlinear science. The localized rational solutions of nonlinear wave equations admit special properties, one of which is that they may have finite critical points (e.g., rational rogue waves) or infinite many critical points (e.g., rational solitons). Rogue waves (a type of rational solitons), originally occurring in the deep ocean [23–26], attracted more and more theoretical and ex-

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perperimental attention in many other fields such as oceanography [27], hydrodynamics [28, 29], plasma physics [30], and nonlinear optics [31–35]. It is believed that rogue waves come from the modulation instability (or Benjamin-Feir instability [36]) and this was noticed by Peregrine [37], see also Ref. [38]. In general, rogue waves can be at least two times higher than the average wave crests and are accompanied by deep troughs, which occurs before and/or after the large crest see Refs. [39, 40] and references therein. Recently, the higher-order rogue wave solutions in nonlinear Schrödinger equation (NLSE) were studied in many articles [41–47]. It is seen that the higher-order rogue waves can be treated as superpositions of several fundamental rogue waves, and the superpositions can create higher amplitudes that still keep located both of space and time. What is more, the hierarchy of rogue wave solutions for other soliton equations have also been reported in references [48–69].

The system of coupled partial differential equations

\[ A_{xx} + \lambda A_{yy} + (u_{xx} + \lambda u_{yy})A + \mu A = 0, \]
\[ u_{xy} = \delta(|A|^2 - C), \delta = \pm 1, \]

where \( \lambda = \pm 1 \), \( A \) is a complex-valued function of \( x, y \), and \( u \) is a real function of \( x, y \). The parameters \( \delta, \mu, \) and \( C \) are freely real constants. This system was introduced by Chow et al. [70], which does indeed bear close resemblance to widely studied model systems in physical applications, e.g. the Davey-Stewartson (DS) equations for water waves in fluid mechanics [70–73]. The fundamental rogue waves and first-order \( W \)-shaped rational solitons have been investigated in Ref. [70]. However, to the best of authors’ knowledge, high-order rogue wave solutions and high-order \( W \)-shaped rational solitons to the coupled system (1) have not been reported before.

In this paper, general representations of two types of rational solutions to the coupled system (1), namely rogue waves and \( W \)-shaped rational solitons, are derived by using the tau-functions of KP hierarchy. The organization of this paper is as follows. In Sec. 2, the main theorem, which is used to express rogue waves and \( W \)-shaped solitons of the coupled system (1) by order-\( N \) determinants, is presented with the help of the bilinear transformation method. The explicit forms and dynamics of rogue wave solutions are discussed in Sec. 3. Typical dynamics of \( W \)-shaped rational solitons are illustrated graphically in Sec. 4. The summary is given in Sec. 5.

2. DERIVATION OF RATIONAL SOLUTIONS OF THE COUPLED SYSTEM

In this Section, we derive the general formulae for \( N \)th-order rational solutions to the coupled system (1). The basic idea is to treat equation (1) as a reduction of the KP hierarchy [18, 74]. Then \( N \)th-order rational solutions of equation (1) can be obtained from rational solutions of the KP hierarchy under this reduction.
Equation (1) is transformed into the following bilinear form:

\[
(D_x^2 + \lambda D_y^2)g \cdot f = 0,
\]

\[
D_x D_y f \cdot f = 2 \delta (g^* - f^2),
\]

through the dependent-variable transformation

\[
A = \sqrt{\frac{2g}{f}}, \quad u = (2 - C)\delta xy - \frac{\mu}{2}x^2 + 2\log f,
\]

where \(f, g\) are functions of the variables \(x, y\), and satisfying the condition

\[
f^*(x, y) = f(x, y),
\]

and the operator \(D\) is the Hirota’s bilinear differential operator \([15]\) defined by

\[
P(D_x, D_y, D_t, \ldots) F(x, y, t, \ldots) \cdot G(x, y, t, \ldots) = P(\partial_x - \partial_{x'}, \partial_y - \partial_{y'}, \partial_t - \partial_{t'}, \ldots) F(x, y, t, \ldots) G(x', y', t', \ldots)\mid_{x' = x, y' = y, t' = t},
\]

where \(P\) is a polynomial of \(D_x, D_y, D_t, \ldots\).

To derive rogue waves in the system (1), we first review rational solutions for equations in the KP hierarchy.

**Lemma 1.** The bilinear equations in the KP hierarchy:

\[
(D_{x_1}^2 - D_{x_2}) \tau_{n+1} \cdot \tau_n = 0,
\]

\[
(D_{x_{-1}}^2 + D_{x_{-2}}) \tau_{n+1} \cdot \tau_n = 0,
\]

\[
(D_{x_{-1}} D_{x_{-2}} - 2) \tau_n \cdot \tau_n = -2 \tau_{n+1} \tau_{n-1},
\]

have rational solutions

\[
\tau_n = \det_{1 \leq i, j \leq N} (m^{(n)}_{ij}),
\]

with the matrix element \(m^{(n)}_{ij}\) satisfying the following differential and difference relations,

\[
\partial_{x_1} m^{(n)}_{ij} = \psi^{(n)}_i \phi^{(n)}_j,
\]

\[
\partial_{x_2} m^{(n)}_{ij} = \psi^{(n+1)}_i \phi^{(n)}_j + \psi^{(n)}_i \phi^{(n-1)}_j,
\]

\[
\partial_{x_{-1}} m^{(n)}_{ij} = -\psi^{(n-1)}_i \phi^{(n+1)}_j,
\]

\[
\partial_{x_{-2}} m^{(n)}_{ij} = -\psi^{(n-2)}_i \phi^{(n+1)}_j - \psi^{(n-1)}_i \phi^{(n+2)}_j,
\]

\[
m^{(n+1)}_{ij} = m^{(n+1)}_{ij} + \psi^{(n)}_i \phi^{(n+1)}_j,
\]

\[
\partial_{x} \psi_i = \psi^{(n+v)}_i,
\]

\[
\partial_{x} \phi_j = -\phi^{(n-v)}_j \quad (v = -2, -1, 1, 2).
\]

These differential and difference relations can be proved by the same method as for Lemma 3.1 in Ref. [47], thus its proof is omitted here. To construct rational
solutions for the bilinear equations (5), we choose functions $m_{ij}^{(n)}$, $\psi_i^{(n)}$ and $\phi_j^{(n)}$ as the following formula

$$
\psi_i^{(n)} = A_i p_i^n e^{\xi_i},
$$

$$
\phi_j^{(n)} = B_j (-q_j)^{-n} e^{\eta_j},
$$

$$
m_{ij}^{(n)} = A_i B_j \frac{1}{p+q} (-\frac{p}{q})^n e^{\xi_i + \eta_j},
$$

where

$$
A_i = \sum_{k=0}^{i} \frac{a_k}{(i-k)!} (p\partial_p)^{i-k}, \quad B_j = \sum_{l=0}^{j} \frac{b_l}{(j-l)!} (q\partial_q)^{j-l},
$$

$$
\xi = \frac{1}{p^2} x - 2 + \frac{1}{p} x - 1 + px_1 + p^2 x_2,
$$

$$
\eta = \frac{1}{q^2} x - 2 + \frac{1}{q} x - 1 + qx_1 - q^2 x_2.
$$

For simplicity, the functions $m_{ij}^{(n)}$ can be rewritten as

$$
m_{ij}^{(n)} = e^{\xi + \eta} (-\frac{p}{q})^n \sum_{k=0}^{n_i} \frac{a_k}{(i-k)!} (p\partial_p + \xi + n)^{n_i-k} \sum_{l=0}^{n_j} \frac{b_l}{(j-l)!} (q\partial_q + \eta - n)^{n_j-l} \frac{1}{p+q},
$$

where

$$
\xi' = -\frac{2}{p^2} x - 2 + \frac{1}{p} x - 1 + px_1 + 2p^2 x_2, \quad \eta' = \frac{2}{q^2} x - 2 + \frac{1}{q} x - 1 + qx_1 - 2q^2 x_2.
$$

Here $p, q, a_k, b_l$ are arbitrary complex constants, and $i, j, n_i, n_j$ and $N$ are arbitrary positive integers.

Further, when taking parameter constraints

$$
q = p^* = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i,
$$

then the reduction condition can be satisfied

$$
(D_{x_2} - D_{x_{-2}}) \tau_{n+1} \cdot \tau_n = 0.
$$

Besides,

$$
(D_{x_1}^2 - D_{x_2}) \tau_{n+1} \cdot \tau_n + (D_{x_{-1}}^2 + D_{x_{-2}}) \tau_{n+1} \cdot \tau_n = 0,
$$

Hence solutions $\tau_n$ also satisfy the bilinear equation

$$
(D_{x_1}^2 + D_{x_{-1}}^2) \tau_{n+1} \cdot \tau_n = 0,
$$

which is the bilinear equation (2) when one takes $\lambda = 1$.

On the other hand, if taking parameter constraints

$$
q = p^* = 1,
$$

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the reduction condition can be met
\[(D x_2 + D x_{-2})\tau_{n+1} \cdot \tau_n = 0. \quad (15)\]

Besides,
\[(D^2 x_1 - D x_2)\tau_{n+1} \cdot \tau_n + (D^2 x_{-1} + D x_{-2})\tau_{n+1} \cdot \tau_n = 0. \quad (16)\]

Hence solutions $\tau_n$ also satisfy the bilinear equation
\[(D^2 x_1 - D^2 x_{-1})\tau_{n+1} \cdot \tau_n = 0, \quad (17)\]

which is the bilinear equation (2) when one takes $\lambda = -1$.

Assuming $x_1, bx_{-1}$ are real, and $x_2, x_{-2}$ are imaginary, we have
\[m_{ij}^*(n) = m_{ij}(-n), \tau^* = \tau_{-n}. \quad (18)\]

Applying the change of independent variables
\[x_1 = x, x_{-1} = -\delta y \quad (19)\]

and taking
\[f = \tau_0, g = \tau_1, g^* = \tau_{-1}, \quad (20)\]

the bilinear equations (5) are reduced to the bilinear equations of the coupled system (2) under parameter constraints (10) and (14). Finally using the gauge freedom of $\tau_n$, general high-order rogue waves of the coupled system defined in (1) can be presented in the following theorem.

**Theorem 1.** The coupled system (1) has $N$th-order rational solutions

\[A = \sqrt{2} \frac{\tau_1}{\tau_0}, u = (2 - C)\delta xy - \frac{\mu}{2} x^2 + 2\log(\tau_0), \quad (21)\]

where
\[\tau_n = \det_{1\leq i,j\leq N}(m_{2i-1,2j-1}^{(n)}), \quad (22)\]

with the matrix elements defined by
\[m_{i,j}^{(n)} = \sum_{k=0}^{n_i} \frac{a_k}{(i-k)!} (p\partial_p + \xi^* + n)_{n_i-k} \sum_{l=0}^{n_j} \frac{a_l}{(j-l)!} (p^*\partial_{p^*} + \xi^* - n)_{n_j-l} \frac{1}{p + p^*}, \quad (23)\]

and
\[\xi^* = px + \delta y. \quad (24)\]

Here $p = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ when taking $\lambda = 1$ in equation (1), and $p = 1$ when $\lambda = -1$. $i, j$ are arbitrary positive integers, and $a_k$ is an arbitrary complex constant.

Similar to the analysis of NLS equation [47], we may also set $a_0 = b_0 = 1; a_2 = a_4 = a_6 = \cdots = b_2 = b_4 = \cdots = 0$ without loss of generality and remain the irreducible complex parameters $a_3, a_5, \cdots, a_{2n-1}$. Furthermore, these rational solutions could also be expressed in a more explicit form in terms of Schur polynomial.
Remark. When taking \( \lambda = 1 \) in equation (1) and \( p = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \) in equation (23), the corresponding rational solutions are \( N \)-th-order rogue wave solutions to the coupled system (1). When taking \( \lambda = -1 \) in equation (1) and \( p = 1 \) in equation (23), the corresponding rational solutions are \( N \)-th-order \( W \)-shaped rational solitons to the coupled system (1).

3. DYNAMICS OF ROGUE WAVE SOLUTIONS

In this Section, we discuss the dynamics of these general rogue wave solutions, which can be obtained by taking \( \lambda = 1, p = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \) in Theorem 1.

We first consider the first-order rogue wave solution. To this end, we set \( N = 1, \lambda = 1, p = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \) in Theorem 1. After a shift of time and space coordinates, \( a_1 \) can be eliminated, hence we set \( a_1 = 0 \). In this case,

\[
m^{(0)}_{11} = (p_1 \partial_p + \xi' + a_1)(p^* \partial_{p^*} + \xi'^* + a_1^*) \frac{1}{p + p^*} |_{p = \sqrt{2} + \sqrt{2} i} = \frac{1}{p + p^*} [ (\xi' - \frac{p}{p + p^*} + a_1)(\xi'^* - \frac{p^*}{p + p^*} + a_1^*) + \frac{pp^*}{(p + p^*)^2} ] |_{p = \sqrt{2} + \sqrt{2} i} = \frac{\sqrt{2}}{2} (x^2 + y^2 - \sqrt{2}x + 1),
\]

\[
m^{(1)}_{11} = (p_1 \partial_p + \xi' + 1 + a_1)(p^* \partial_{p^*} + \xi'^* - 1 + a_1^*) \frac{1}{p + p^*} |_{p = \sqrt{2} + \sqrt{2} i} = \frac{1}{p + p^*} [ (\xi' + 1 - \frac{p}{p + p^*} + a_1)(\xi'^* - 1 - \frac{p^*}{p + p^*} + a_1^*) + \frac{pp^*}{(p + p^*)^2} ] |_{p = \sqrt{2} + \sqrt{2} i} = \frac{\sqrt{2}}{2} (x^2 + y^2 - \sqrt{2}x - \sqrt{2}iy + i).
\]

Hence the first-order rogue wave is

\[
A = \sqrt{2} [ 1 + \frac{2i\delta y - 2ix + \sqrt{2}i - \sqrt{2}}{x^2 + y^2 - \sqrt{2}x + 1} ],
\]

\[
u = (2 - C)\delta xy - \frac{\mu}{2}x^2 + 2\log\left(\frac{\sqrt{2}}{2}(x^2 + y^2 - \sqrt{2}x + 1)\right).
\]

The square of the short wave amplitude \( |A|^2 \) possesses four critical points, namely

\[
A_1 = \left( \frac{\sqrt{2} + 1}{2}, -\frac{1}{2\delta} \right), A_2 = \left( \frac{\sqrt{2} - 1}{2}, \frac{1}{2\delta} \right), A_3 = \left( \frac{\sqrt{2} + 1}{2}, \frac{1}{2\delta} \right), A_4 = \left( \frac{\sqrt{2} - 1}{2}, -\frac{1}{2\delta} \right).
\]

Based on the analysis of critical points for rogue-wave solutions (25), there are four-petaled rogue waves (i.e., two global maximum points \( A_1, A_2 \), and two global minimum points \( A_3, A_4 \)) in the coupled system (1). The maximum value of \( |A| \) is 2 at
points $A_1$ and $A_2$, while the the minimum value of $|A|$ is 0 at the points $A_3$ and $A_4$. This fundamental rogue wave is illustrated in Fig. 1 with parameters $\delta = 1$.

High-order rogue waves can be derived by taking $N > 1, \lambda = 1, p = 7 \sqrt{2} + i \sqrt{7}$ in Theorem 1. The $N$th-order rogue wave is composed of $\frac{N(N+1)}{2}$ individual fundamental rogue waves, and the interaction between these fundamental rogue waves can generate various patterns of $(1 + 1)$-dimensional rogue waves, such as fundamental patterns, triangular patterns and circular patterns.

To demonstrate high-order rogue waves, we first consider the second-order rogue waves, which can be derived by taking $N = 2, \lambda = 1, p = 7 \sqrt{2} + i \sqrt{7}$ in Theorem 1. In this case, with parameter choices $a_1 = a_2 = 0$, the second-order rogue wave reads

$$A = \sqrt{2}(1 + \frac{\phi}{f_2}), u = (2 - C)\delta xy - \frac{\mu}{2} x^2 + 2\log f_2,$$

(26)

where

$$f_2 = \frac{7}{4} x^2 + \frac{3}{4} y^2 + \frac{7}{16} \sqrt{3} x y^2 - \frac{1}{4} \sqrt{3} x^3 y^2 + \frac{1}{36} y^6 - \frac{1}{2} \sqrt{2} x^2 y - \frac{1}{24} \sqrt{3} x^3 y + \frac{13}{48} y^4 + \frac{29 x^4}{48} + \frac{1}{3} x^6 - \frac{1}{12} \sqrt{2} x^3 y^2 + (-\frac{1}{8} - \frac{1}{8} \sqrt{3} x y^2 - \frac{1}{4} xy - \frac{i}{24} \sqrt{3} x^3 - \frac{i}{24} \sqrt{3} y^3 + \frac{1}{4} y^2$$

$$- \frac{1}{4} x^2 + \frac{1}{4} \sqrt{2} x + \frac{1}{24} \sqrt{2} x^3 + \frac{i}{8} \sqrt{2} x y^2 - \frac{i}{8} y^2 - \frac{1}{24} \sqrt{2} y^3 + \frac{i}{8} x^2 + \frac{i}{8} \sqrt{2} x y + \frac{1}{24} \sqrt{2} y^3$$

$$- \frac{1}{2} x y + \frac{i}{2} \sqrt{2} x y a_1 - \frac{1}{8} \sqrt{2} x y^4 + (-\frac{1}{8} - \frac{1}{8} \sqrt{2} x y^2 - \frac{1}{4} xy + \frac{1}{24} \sqrt{2} x^3 + \frac{i}{24} \sqrt{3} y^3$$

$$+ \frac{1}{4} y^2 + \frac{1}{4} x^3 - \frac{1}{4} x^2 + \frac{1}{4} \sqrt{2} x + \frac{1}{24} \sqrt{2} x^3 + \frac{i}{8} \sqrt{3} x y^2 + \frac{i}{8} y^2 - \frac{1}{24} \sqrt{2} y^3 + \frac{i}{8} x^2$$

$$- \frac{i}{8} \sqrt{2} x y^2 - \frac{i}{4} \sqrt{2} y + \frac{i}{2} x y + \frac{1}{8} \sqrt{3} x y a_3 - \frac{1}{24} \sqrt{2} y^3 + \frac{1}{12} x^2 y^4 - \frac{1}{3} \sqrt{2} y^3$$

$$+ \frac{1}{3} x^3 y^2 + \frac{1}{8} \sqrt{2} y + \frac{2}{12} x^3 y^2 + \frac{1}{2} x y + \frac{1}{3} x^3 y - \frac{\sqrt{2} x^2 y}{8} - \frac{11 x^2 y^3}{12} - 1/8 \sqrt{2} x^5,$$

$$\phi = \frac{5}{2} \sqrt{2} x^2 - \frac{3}{2} \sqrt{2} y^2 - \frac{1}{4} \sqrt{3} x y - \frac{i}{12} \sqrt{3} x y^4 - \frac{1}{2} \sqrt{2} x y^2 + \frac{2}{12} x^2 y^2 + \sqrt{2} y - \frac{3}{4} x^2 y - \frac{1}{3} \sqrt{2} x y^4 + \frac{\sqrt{2} x^2 y}{6} + \frac{5}{6} \sqrt{2} x^3 y +$$

$$\sqrt{2} x^2 y - \frac{3}{4} + \frac{1}{4} x^2 y - \frac{i}{4} \sqrt{2} x y^2 - \frac{i}{4} x^2 y - \frac{i}{4} \sqrt{2} x^3 y + \frac{1}{4} x y + \frac{i}{4} \sqrt{2} x y,$$

$$+ \frac{i}{3} \sqrt{2} y - \frac{i}{4} x^2 + \frac{1}{4} \sqrt{3} x y - \frac{i}{2} x y + \frac{i}{2} \sqrt{2} x y a_3 - \frac{i}{12} \sqrt{2} y^3 + \frac{2}{3} x y^3 - \frac{i}{12} \sqrt{2} y^3 - i \sqrt{2} x^2 y -$$

$$i x y - \frac{3}{4} \sqrt{2} x + \frac{2}{3} i x^2 y + i x y a_3 - \frac{7}{6} i \sqrt{2} x^3 - \frac{i}{12} \sqrt{2} x^5 - i \sqrt{2} x^3 y - i \sqrt{2} x^2 y,$$

where $a_3$ is a free complex parameter. This solution is displayed in Fig. 2. As can be seen, this solution contains three fundamental rogue waves. Selecting different values of parameter $a_3$, different patterns of second-order rogue waves can be generated, such as fundamental patterns (see Fig. 2(a)) and triangular patterns (see Fig. 2(b)).
Besides, for other selection of $a_3$, the locations of these three fundamental rogue waves can be changed, hence some other patterns of second-order rogue waves may be fund. It is noticed that for all the choices of parameters $a_3$, the maximum value of the solution $|A|$ does not exceed $2\sqrt{2}$ (i.e., two times the constant background). Thus this interaction between the three fundamental rogue waves does not generate very high peaks in the coupled system (1).

![Fig. 1 – First-order rogue waves (25) with parameter $\delta = 1$. The right panel shows the density of the left pattern.](image1)

![Fig. 2 – Second-order rogue waves (26) with parameters: (a) $a_3 = \frac{1}{20}, \delta = 1$; (b) $a_3 = 20/3, \delta = 1$.](image2)

![Fig. 3 – Third-order rogue waves with parameters: (a) $a_3 = 1/20, a_5 = 1/200, \delta = 1$; (b) $a_3 = 30i, a_5 = 0, \delta = 1$; (c) $a_3 = 0, a_5 = 200i, \delta = 1$.](image3)

For larger $N$ in Theorem 1, these higher-order rogue waves have qualitatively similar behaviors, except that more fundamental rogue waves would interact with each other, and more complicated wave fronts will form in the interaction region. For example, with $N > 1, \lambda = 1, p = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and different parameter choices of $a_3$ and $a_5$, different patterns of third-order rogue waves would be generated, which are shown in Fig. 3. It is seen that the third-order rogue wave in the coupled system (1) are composed of six four-petaled rogue waves. The interaction of there six
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four-petaled rogue waves can generate different types of second-order rogue waves, such as fundamental patterns (see Fig. 3(a)), triangular patterns (see Fig. 3(b)), and circular patterns (see Fig. 3(c)). But again, the maximum value of this solution $|A|$ stays below $2\sqrt{2}$ (i.e., two times the constant background) for all the selection of $a_3, a_5$, so this interaction does not create very high spikes either. Note that in the NLS equation, the maximum value of third-order rogue wave is equal to seven times the constant background. Thus the third-order rogue waves in the coupled system (1) do not generate high peaks as rogue waves in the NLS equation, and such rogue waves are also dangerous if they arise in physical situations.

4. DYNAMICS OF $W$-SHAPED SOLITON SOLUTIONS

In this Section, we focus on dynamics of $W$-shaped rational soliton solutions to the coupled system (1), which can be obtained by taking $\lambda = -1, p = 1$ in Theorem 1. In this case, the corresponding $N$th-order rational solutions describe interactions between $N$ parallel $W$-shaped line waves.

Taking $N = 1, \lambda = -1, p = 1$ in Theorem 1, the first-order $W$-shaped rational soliton solution can be expressed as

$$A = \sqrt{2}\left[1 + \frac{1}{(x + \delta y - \frac{1}{2} + a_1)^2 + \frac{1}{4}}\right],$$

$$u = (2 - C)\delta xy - \frac{\mu}{2}x^2 + 2\log\left[(x + \delta y - \frac{1}{2} + a_1)^2 + \frac{1}{4}\right],$$

where $a_1$ is a freely complex constant. It is not hard to find that $|A|$ is a line soliton, see Fig. 4. Based on the analysis of critical lines for rational solutions (27), the maximum value of $|A|$ is $3\sqrt{2}$ (i.e., three times the constant background amplitude) at the center $(x + \delta y = 0)$ of the line wave with $a_1 = 0$ (see Fig. 4(c)). Specially, $|A| \to 0$ when $|a_1| \to \pm \infty$. This fundamental $W$-shaped line soliton is illustrated in Fig. 4 with different choices of $a_1$.

![Fig. 4 – First-order $W$-shaped rational soliton solutions (27) with parameters $\delta = 1$ and different choices of $a_1$: (a) $a_1 = 10i$, (b) $a_1 = 1$, (c) $a_1 = 10$.](image)

For an arbitrary given value of $N$, we can obtain the $N$th-order $W$-shaped rational soliton solutions by employing the results of Theorem 1. For example, setting $N = 2, \lambda = -1, p = 1$ in Theorem 1 yields the following explicit form of the second-
order $W$-shaped rational soliton solutions:

$$A = \sqrt{2}(1 + \frac{\phi}{f'}), u = (2 - C)\delta xy - \frac{H}{2}x^2 + 2\log f', \quad (28)$$

where

$$f' = \left(\frac{1}{32} \delta y^2 - \frac{1}{48} \delta y^3 - \frac{1}{16} \delta y^2 x - \frac{1}{16} \delta x^3 y + \frac{1}{16} x \delta y + \frac{1}{48} \delta x^2 y - \frac{1}{48} x^3 + \frac{1}{16} x \delta y + \frac{1}{48} a_3 \right) a_3 - \frac{x}{64} + \frac{1}{32} x^2$$

Here we have taken $a_1 = a_2 = 0$, and $a_3$ is an arbitrary complex parameter.

The evolution of solution $|A|$ given by equation (28) is plotted in Fig. 5. It can be seen that, when $|a_3| \to \infty$, this solution $|A|$ approaches to the constant background, namely $|A| \to 0$. When $a_3 = -\frac{1}{17}$, $|A|$ reaches the maximum values $5\sqrt{2}$ (i.e., five times the height of the background); its profile is shown in panel (c) of Fig. 5.

![Fig. 5 - Second-order $W$-shaped rational soliton solutions (28) for the coupled system (1) with parameter $\delta = 1$ and different choices of $a_3$: (a) $a_3 = 25i$, (b) $a_3 = 10$, (c) $a_3 = -\frac{1}{17}$](image)

For larger $N$ and $\lambda = -1, p = 1$ in Theorem 1, these high-order rational solutions have qualitatively similar behaviors, except that more $W$-shaped rational solitons interact with each other, and more complicated wave fronts would form in the interaction region. For instance, for $N = 3, \lambda = -1, p = 1$ and parameter choices

$$a_0 = 1, a_1 = 0, a_2 = 0, a_4 = 0, \quad (29)$$

and different choices of $a_3$ and $a_5$, the corresponding solution is shown in Fig. 6. As can be seen, the corresponding solution is composed of three parallel $W$-shaped
line waves (see Fig. 6(a)). The maximum value of this solution $|A|$ is equal to $7\sqrt{2}$ (i.e., seven times the height of the background) for $a_3 = -\frac{1}{12}, a_5 = \frac{1}{240}$; its profile is shown in panel (c) of Fig. 6.

Motivated by the key features of the line waves for the DSIII equation [75], the obtained results support the following conjectures for the $N$th-order $W$-shaped solitons $|A|$ of equation (21):

1. The $N$th-order $W$-shaped rational solitons are composed of $N$ individual $W$-shaped line waves.

2. For the fundamental pattern where the height of the background is $\sqrt{2}$, the central peak is surrounded by $n(n+1) - 2$ gradually decreasing peaks. The maximum value of $|A|$ is equal to $\frac{n(n+1)\sqrt{2}}{2}$ (i.e., $\frac{n(n+1)}{2}$ times the height of the background).

5. SUMMARY AND DISCUSSION

In conclusion, two types of general high-order rational solutions to the coupled system (1), namely rogue waves and $W$-shaped solitons, are investigated by utilizing the bilinear method, which are expressed in terms of determinants. The basic idea is to treat the coupling system (1) as constrained KP hierarchy. Then, we derive rational solutions of the coupled system (1) from rational solutions of the KP hierarchy, which are given in a simple representation of the Grammian determinants. Moreover, we analyze in detail the features and asymptotic behavior of the high-order rogue waves up to third-order rogue waves. Besides, some superposition patterns have displayed for different choices of parameters, see Figs. 1, 2, and 3. The typical dynamics of these $W$-shaped solitons are also illustrated graphically, see Figs. 4, 5, and 6. Our solutions contribute to better controlling and understanding of rogue wave phenomena in a variety of complex dynamics, ranging from fluid dynamics, to optical communications, Bose-Einstein condensates, and financial systems. Finally, it is worthy to emphasize that the technique presented in this paper may also succeed to other multi-dimensional nonlinear integrable systems, and we also expect that new
features of rogue waves solutions will be identified using this technique. The related results will be reported elsewhere.

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