USING EXCEL SPREADSHEETS TO STUDY THE VIBRATING STRING BEHAVIOR

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Abstract. Using Excel spreadsheets, we developed a simple simulation software that simulate the vibration of a string and designed to be used in the physics teaching process. Using a special devised animation, the tool will help students to understand and, especially, to visualize the solutions of the equation of the oscillating string. We devised two spreadsheets, one for the plucked and one for the strucked string. We used this tool to analyze the vibration modes of the string and to investigate the way that it may be used in increasing the effectiveness of teaching.

Key words: vibrating string, Excel spreadsheets, fundamental frequency, plucked string, strucked string, normal modes, simulation, teaching

1. INTRODUCTION

During the years, experience showed that computer simulation and use of the IT technologies play an important role in teaching physics [1, 2]. Even from the advent of the computer technology, scientist starting to use this new tool in physics, first for computer simulations in many domains of physics [3-6], especially in the condensed state physics [7-16], or as effective aids for doing theoretical calculations [17]. In the last years, computers were used at physics lessons not only for processing data [18], or for e-learning purposes [19, 20], but also as tools for doing virtual physics experiments and efficient modelling tools [21, 22].

Within the current framework, Excel spreadsheets become valuable educational tools in teaching physics, due to its versatility, ease of use and relatively large scale use [24-26]. On the other hand, researcher in the domain of Physics Education consider that a visual tool like an Excel table, along with its capability of data presentation, help students to increase their capacity of understanding concepts of physics [27-29]. Of course, there are many other commercial simulation tools that, when are appropriate used, may increase the interest of students in learning physics [28-33]. There are also many reports about specially devised software programs that may be used in teaching physics [23-26, 34-37].

It is also considered that this kind of simulations supports students with cognitive restrains to gain a better understand of physics [23], because the Excel packet is very user friendly.

In the present paper, we used Excel spreadsheets to simulate the behavior of the vibrating string. The vibrating string is a very promising study domain for both

theoretical and experimental point of view for every physicist. It has many immediate and important applications: it is an important model used in teaching physics of the travelling waves, but also has many practical uses in numerous engineering systems, like ropes, belts, power transmission lines, or wire transportation systems [38].

On the other hand, the study of the vibrating string is a very challenging task for students of the first physics grade. The difficulties arise not only for derivation of the equation of motion, but especially at the solving it, because it is a partial derivatives equation of the second grade [38-41].

From our teaching experience, many students focus on solving the wave equation, leaving aside the interpretations of the solution.

Having this in mind, we developed a very simple tool made in Excel to help students to understand and, especially, to visualize the solutions of the equation of the oscillating string. Using the Excel spreadsheets, we made a spatial and temporal discretization of the solution of the wave equation, along with a visual tool that allows an easy visualization of the normal modes of vibration.

We considered two particular situations: the so-called plucked string, when a certain point of the string is taken out from its equilibrium state by an arbitrary distance and released with initial zero velocity [40]. The second one is the strucked string, when the string is not initially displaced, but gain an initial velocity suddenly applied on some portion of it by means of some kick [38].

Because there is a relatively large number of parameters that determine the behavior of the string, our Excel spreadsheet may be a valuable tool when exists the necessity of investigate the role of some parameters in behavior of the string.

In the remaining of the paper, we will describe the mathematical model of the vibrating string, along with its solutions. Using this equation, we will explain the construction of the Excel spreadsheets and we will give some results of the simulation, in order to present its capabilities.

2. MATHEMATICAL MODEL AND SIMULATION METHOD

2.1 MATHEMATICAL MODEL

In the first part of this section, we will analyze the mathematical model of the vibrating string, in order to identify parameters that are necessary to perform the simulations.

We consider a homogeneous infinite thin string of length L, laying in the OX direction and fixed at both ends: $x = 0$ and $x = 7$, as in Figure 1. Let T be the tension throughout the string and μ its linear mass density. It is assumed that the weight of the rope is small compared to the tension and may be neglected.

Fig. 1 Infinitesimal portion of a spring supposed to a small displacement in the transverse direction.

In this situation, it may be proved that that there are not longitudinal displacements and the string vibrates in the XOY plane, [40], its movement being described by the function $u(x,t)$ that describe the motion of each point of the vibrating string at any time moment along the OY axis.

According to [38-41], equation of motion of the vibrating string is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

where

$$v = \sqrt{\frac{T}{\mu}} \tag{2}$$

is the velocity which the wave propagates along the string. It depends only on tension and linear mass density that is on the intrinsic properties of the rope.

Equation (1) is a second order partial differential equation for the function $u(x,t)$, which describe the motion of each point of the vibrating string at any time moment [41].

It is supposed that, at $t = 0$, function $u(x,t)$ satisfies the initial conditions:

$$u(x,0) = u_0(x) \tag{3}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \dot{u_0}(x) \tag{4}$$

The above relations assume that the initial positions and velocities of each point of the string are known.

We study the case where both ends of the string are fixed, that is the solutions of wave equation have to satisfy the boundary conditions:

$$u(0,t) = 0 \tag{5}$$

$$u(L,t) = 0 \tag{6}$$

The vibrating string problem, *i.e.* partial differential equation (1) with initial conditions (3) and (4) and boundary conditions (5) and (6), can be solved using the separation of variables method [38-41], its general solution being:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(A_n \cos\omega_n t + B_n \sin\omega_n t) \qquad (7)$$

the constants are determined from the initial conditions (3) and (4):

$$A_n = \frac{2}{L}\int_0^L u_0(x)\sin\frac{n\pi x}{L}dx \qquad (8)$$

$$B_n = \frac{2}{n\pi v}\int_0^L \dot{u}_0(x)\sin\frac{n\pi x}{L}dx \qquad (9)$$

The general solution (7) is obtained as a linear combination of normal modes of vibration [40]:

$$\omega_n = 2\pi v_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}, \qquad n = 1, 2, 3, ... \qquad (10)$$

From equation (10), it may be seen that normal frequencies of oscillation are independent of position, being the same for all points of the string. Also, we notice that the normal frequencies increase with the velocity, thus increase with tension and decrease with the linear mass density.

On the other hand, the amplitude of the oscillation depends on position for each mode of the string:

$$A_n = \sqrt{A_n^2 + B_n^2}\left|\sin\frac{n\pi x}{L}\right| \qquad (11)$$

The maximum amplitude $\sqrt{A_n^2 + B_n^2}$ is obtained for $\sin\frac{n\pi x}{L} = \pm 1$.

2.2 SIMULATION METHOD

The most used ways of producing initial excitation of a vibrating string are the plucked string method and the struck string method, both of them giving initial positions and velocities for the points of the vibrating string [40]. Positions and velocities of all point of the string are functions $u_0(x)$ and $\dot{u}_0(x)$ which are used to evaluate the integrals from (8) and (9), [40].

On a plucked string, a point situated at distance d from one of its ends is taken out from its equilibrium state by a distance h and released with initial zero velocity.
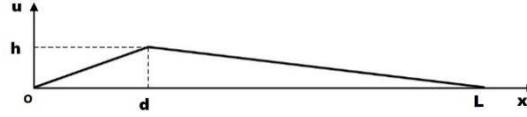
Fig. 2 Plucked string initial conditions.

If it is assumed the contiguity of the string and zero boundary conditions as in Figure 2, it is reasonable to consider a linear dependence of the disturbance on distance, the mathematical expression of the initial conditions of the string points being:

$$u_0(x) = \begin{cases} \dfrac{hx}{d}, & 0 \le x \le d \\ \dfrac{h(L-x)}{L-d}, & d < x \le L \end{cases} \tag{12}$$

$$\dot{u}_0(x) = 0, x \in [0, L] \tag{13}$$

Using condition (12) and (13), the Fourier coefficients are obtained by a tedious, but elementary integration:

$$A_n = \frac{2hL^2}{\pi^2 n^2 d(L-d)} \sin\frac{n\pi d}{L} \tag{14}$$

$$B_n = 0 \tag{15}$$

and the solution of the wave equation for the plucked string is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2hL^2}{\pi^2 n^2 d(L-d)} \sin\frac{n\pi d}{L} \sin\frac{n\pi x}{L} \cos\frac{vn\pi}{L} t \tag{16}$$

The idea behind calculation in Excel of the vibration modes of the string is to calculate each term in the equation (16), in other words, its vibrations modes. First of all, it is necessary to give values to the parameters of the string.

For ease of use, we give to the cell very descriptive names. This way of naming is very appropriate not only for the developer point of view, but especially for the educational purpose of the tool: the student will focus on the physical meaning of the parameters, instead of trying to remember some obscure names of Excel cells.

So, we name the cell accordingly with their use: B1 "Tension", B2 "miu", B3 "v" - velocity of the wave calculated with the Excel formula "=SQRT(Tension/miu)" based on equation (9), B4 "L" - the length of the string, B5 "d" - the point where the string is displaced, B6 "h" - the value of the displacement. Some typical values are illustrated in Figure 3. For teaching purposes, we putted the measurement units, in the range B1:B6.

	A	B	C
1	Tension	1	N
2	miu	1	kg/m
3	v	1	m/s
4	L	1	m
5	d	0.2	m
6	h	0.1	m

Fig. 3 Values for the parameters of the plucked string.

General solution of the wave equation, relation (7), as well as particular solution of the plucked string, relation (16), describe the elongation in each point of it. So, we take 100 hundred points where u(x, t) will be calculated, resulting a space step denoted in cell B8 by "Delta_x" and calculated as "=L/100". The number of 100 points is taken for illustration purposes only. It is obvious that we may take a finer step by simply changing 100 with, say 1000. Value Delta_x is used to calculate values in column D where are kept the points were solution of the wave equation is evaluated.

The idea behind the Excel calculation is based on the term of vibration modes of the string [38-41]. First of all, it is necessary to calculate the main oscillation mode, the subsequent ones being calculated by varying *n*. So, using the appropriates values defined above, we evaluated parameters from equation (16):

$\dfrac{2hL^2}{\pi^2 d(L-d)}, \dfrac{\pi d}{L}, \dfrac{\pi x}{L}$, and $\dfrac{v\pi}{L}$. These parameters are calculated in the range A13:B35.

The most difficult problem was to put to work the time dependence. In this situation, in cell B9, we defined a time step arbitrarily chosen 0.004. Our tests, found that a bigger one is less accurate and a smaller would give long time calculations.

In order to have time as parameter, we first modified one option in Excel: File -> Option -> Formulas from where we choose Workbook Calculation Automatic, check Enable iterative calculation. Note that this procedure is valid for Microsoft Office Excel™ 2007-2016.

The cell B10 will be set to have a circular reference, by putting its value "=B10 + Delta_t". Pressing F9 key will make calculate the worksheet formulas including the new values for t. A continuous press of F9 key will produce a continuos time iteration.

All the quantities described above, along with the iterated time, are used in column E, where, for n=1 in equation (16), will give elongationsof the first vibration mode in each point of the string and at every moment of time.

We repeat the procedure from column E in columns F, G, H, I, calculating the first five normal modes. In column J, we calculated the total elongation as superposition of the first five normal modes.

Our estimation shows that five modes are enough, because, for the parameters in Figure 3, we found that the amplitude of the first normal mode is 0.081 and for the fifth is 0.003, contribution for higher modes being smaller and smaller.

D	E	F	G	H	I	J
x(m)	u1 (m)	u2(m)	u3(m)	u4(m)	u5(m)	u=u1+...+u5
0	0	0	0	0	0	0
0.01	0.002546	1.56E-19	-0.00085	-1.6E-19	0.000507	0.00220569
0.02	0.00509	3.11E-19	-0.00169	-3.1E-19	0.001002	0.00440391
0.03	0.007628	4.65E-19	-0.00251	-4.6E-19	0.001472	0.00658741
0.04	0.010159	6.17E-19	-0.00332	-6E-19	0.001906	0.00874944
0.05	0.01268	7.67E-19	-0.00409	-7.3E-19	0.002293	0.01088395
0.06	0.015189	9.14E-19	-0.00483	-8.5E-19	0.002623	0.01298578
0.07	0.017682	1.06E-18	-0.00552	-9.6E-19	0.002889	0.01505087
0.08	0.020158	1.2E-18	-0.00617	-1E-18	0.003084	0.01707638
0.09	0.022614	1.33E-18	-0.00676	-1.1E-18	0.003202	0.01906078
0.1	0.025048	1.46E-18	-0.00729	-1.2E-18	0.003242	0.02100398
0.11	0.027457	1.58E-18	-0.00775	-1.2E-18	0.003202	0.0229073

Fig. 4 Values of the elongation for the first five vibration modes and for their summation.

Elongation estimated in columns I-J is for each bin of the string defined for the pass Delta_x and for every time pass defined by Delta_t and may be used for subsequent calculations.

In the case of the strucked string, its points are not initially displaced, but have an initial velocity, suddenly applied on some portion of them. Let say that this velocity is applied at position *d* by a hammer with width *s*.

In this case, the initial conditions of the wave equation are:

$$u_0(x) = 0, \quad x \in [0, L] \tag{17}$$

$$\overset{\bullet}{u_0}(x) = \begin{cases} v_0, & d - \dfrac{s}{2} < x < d + \dfrac{s}{2} \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

Straightforward evaluation of the coefficients from equations (8) and (9), gives us the solution of the wave equation for the strucked string:

$$u(x,t) = \frac{4v_0 L}{\pi^2 n^2 v} \sum_{n=1}^{\infty} \sin\frac{n\pi d}{L} \sin\frac{n\pi s}{2L} \sin\frac{n\pi x}{L} \cos\frac{vn\pi}{L} t \tag{19}$$

The idea behind Excel simulation is the same as before. The parameters of the simulation are defined as follows: B1 "Tension", B2 "miu", B3 "v" - velocity of the wave, again calculated with the Excel formula "=SQRT(Tension/miu)" based on equation (2), B4 "L" - the length of the string, B5 "d" - the point where the center of the hammer smash the string, B6 "s" - the length of the hammer, B7 "v0" velocity imposed by the hammer.

An example of typical values is depicted in Figure 5.

	A	B	C
1	Tension	1	N
2	miu	1	kg/m
3	v	1	m/s
4	L	1	m
5	d	0.3	m
6	s	0.1	m
7	V0	0.2	m

Fig. 5 Values for the parameters of the strucked string.

The rest of the Excel spreadsheet is very similar with the corresponding one of the plucked string, calculating the first five oscillating modes and their summation.

3. RESULTS AND DISCUSSIONS

Using the spreadsheets presented before, we made a series of simulations, in order to test their usability and capacity of being used as teaching tools. Because the large number of parameter that can be varied, the quantity of data is very big.

3.1 SIMULATION OF THE PLUCKED STRING

The first set of simulation we present is for parameters from Figure 3. The string is asymmetrically plucked at d=0.2 for L=1. At the moment t=0, the graphs of the first five normal modes is presented in Figures 6, along with their summation.
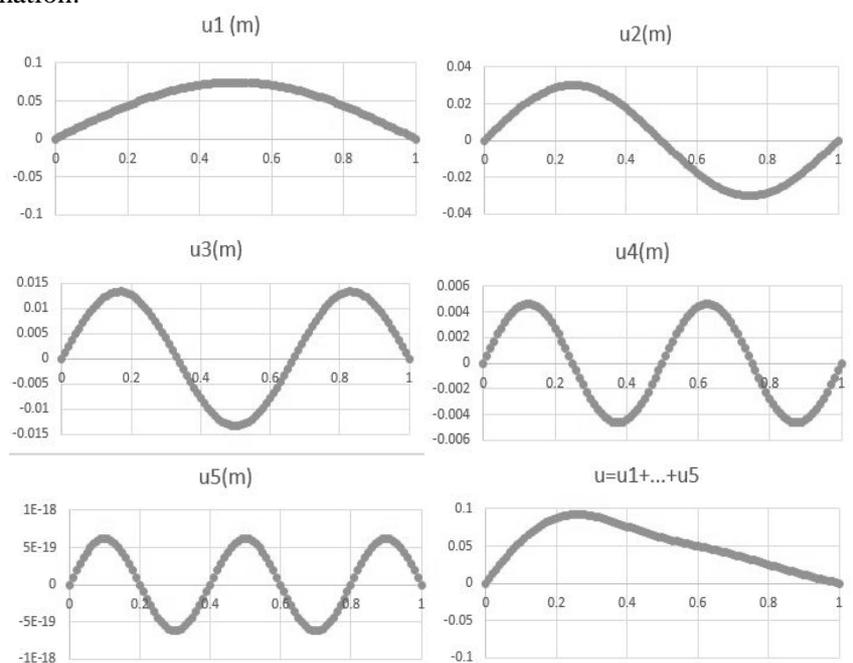
Fig. 6 First five normal modes and their summation of the plucked string at $t = 0$.

By analyzing the spatial dependence of the first five normal modes, we found that, as it is expected, each of it represents a harmonic motion with different wavelength. The last graph in Figure 6 shows the spatial dependence of the string at the beginning of the oscillation, the shape being as expected by imposing the initial conditions of the plucked string with $d = 0.2$ and $L = 1$.

We have to note that the graphs from Figure 6 are only snapshots, because, if we press the F9 key continuously, the graphs are animated. The user may variate the parameters of the string and, from the animation, he immediately will gain a visual feedback about the frequency of oscillation, according to the equation (10).

If we choose $d = \dfrac{L}{2}$, the amplitudes of the even modes vanish, because in this case $\sin\dfrac{n\pi d}{L} = 0$ in equation (16), due to the fact that the argument of the sine is an integer of π. Of course, this may also happen in the all the cases when this argument is integer.

Another analysis we may make using the Excel tool for the plucked string, is the distribution of the amplitudes versus position of the plucking. This kind of analysis is useful for investigating the quality of sounds produced by certain musical instrument having plucked string, like guitars [41, 42].

From equation (11), amplitude of the non-spatial modulated oscillation is:

$$A_n = \frac{2hL^2}{\pi^2 n^2 d(L-d)}\left|\sin\frac{n\pi d}{L}\right| \tag{20}$$

For the first five oscillation modes, the simulated amplitude decrease, as it is predicted by equation (20).

A simulation for $L = 1$ and $d = 0.5$ shows that a musical instrument will produce pure tone if it is plucked near its center, because the secondary harmonics will be very small and eventually vanish, as we may notice from Figure 7. We again sign the totally disappearance of the even oscillations in the case $d = \dfrac{L}{2}$.
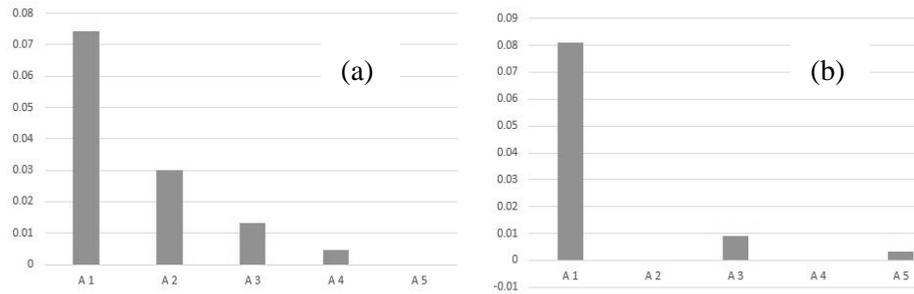

Fig. 7 Distribution of the amplitudes of the first five normal modes of the plucked string with $L = 1$
a) $d = 0.2$; b) $d = 0.5$

The user may freely choose where to pluck the string by setting the Excel parameter *d* and also the initial displacement of the string by setting a value for *h*.

3.2 SIMULATION OF THE STRUCKED STRING

As we discussed before, the idea behind using Excel spreadsheets to simulate the behavior of the strucked string is the same with that of the plucked string. The parameters describing the comportment of the string are again tension, mass density, length, but also the position where the hammer struck the string, its dimension and velocity. In this case, the variability is increased as we may see from equation (19).

We illustrate the results of the simulation of a strucked string using the parameters from Figure 5.
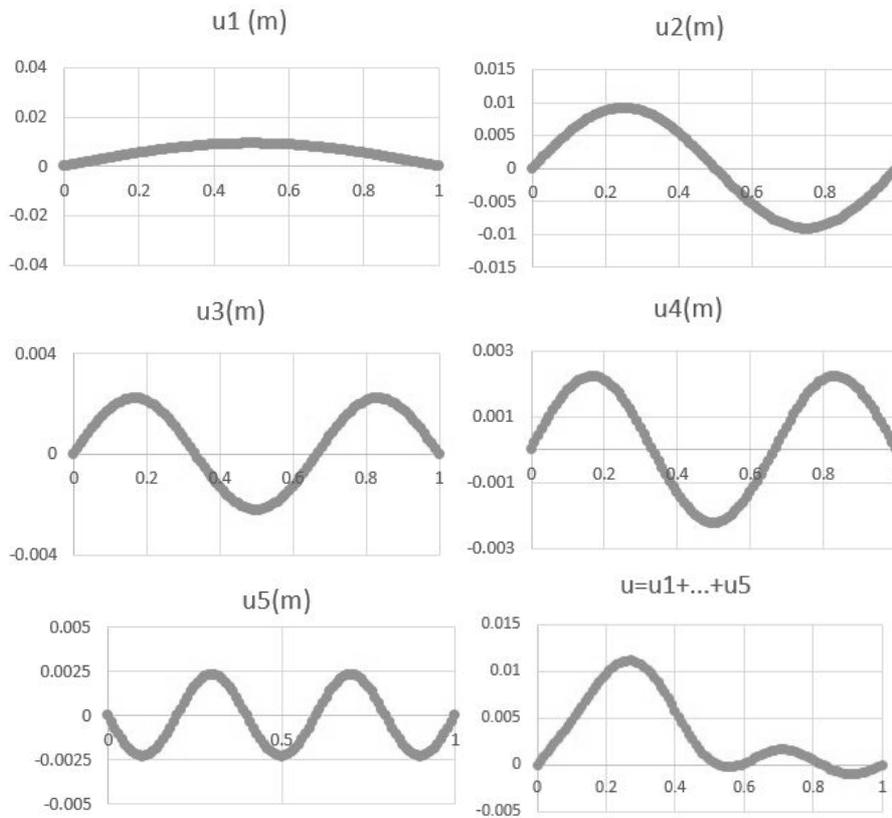
u1 (m)

u2(m)

u3(m)

u4(m)

u5(m)

u=u1+...+u5

Fig. 8 First five normal modes and their summation of the strucked string at $t = 0.15$.

At the beginning of the movement, the initial condition of a plucked string impose that all elongations are zero and we choose another moment of time, smaller than half period of the fifth oscillation mode, which is the smallest one.

As we already discuss before for the plucked string, the graphs from Figure 8 are again snapshots of the oscillations modes of the string which are animated by continuously pressing the F9 key. Once more, the user has the possibility to choose different values for elastic string tension and linear mass density, obtaining the visual feedback from animation.

For both strucked and plucked string, the user has the option to obtain a smoother animation by adjusting the parameter Delta_t in each spreadsheet.

The first five amplitudes of the normal modes are depicted in Figure 9:
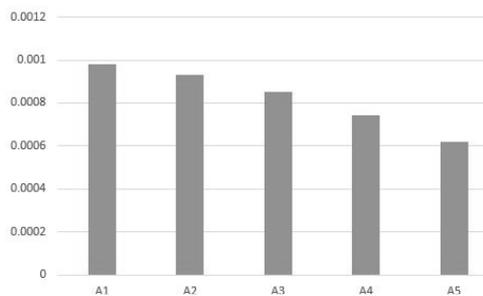
Fig. 9 Distribution of the amplitudes of the first five normal modes of the strucked string

We again found that amplitude decrease with the square of the frequency, as equation (30) predicts.

4. CONCLUSSIONS

We developed a simple tool made in Excel to help students to understand the physics behind the oscillating string. Starting from the solutions of the wave equation, we developed a piece of software that, using spatial and temporal discretization, allows an immediate visualization of the normal modes of vibration of a string. The tool address two modes of imposing vibrations to it: the pluck and the struck, both of them imposing initial conditions that determine the future behavior.

Our program permits an easy change of the parameters of the string, like length, tension, density, position an characteristics of the external initial perturbations, the results, being immediately and visibly available.

The simulation program also may be used to simulate oscillation of a vibrating string for any initial condition, as long as the Fourier coefficient may be calculated.

We succeeded to obtain a simulation of the vibrating string without using the VBA programming language. This make our software safer and it may be widely used on different machines where the security policy maybe deny usage of macros.

On the other hand, usage of only simple features of Excel, without resorting to some difficult programming techniques, make the software very accessible to students which may freely modify it.

The main conclusion of our investigation is that Excel spreadsheets may be successfully used to simulate physical phenomena, being a cheap and very user friendly tool.

The Excel spreadsheets represent an alternative and complementary method of teaching physics. A simple summarization of the large number of the potential experiments we can do with this simple Excel spreadsheet, is evidence that this tool is valuable in the understanding the physics behind the vibrating string, the engineering devices and musical instrument having chords.

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