GENERALIZED MASTER EQUATION, BOHR'S MODEL, AND MULTIPOLES ON FRACTALS

SALEH ASHRAFI[1,a], ALI KHALILI GOLMANKHANEH[1,b], DUMITRU BALEANU[2,3,c]

[1]Tabriz University, Faculty of Physics, Tabriz, Iran
Email:[a] *ashrafi@tabrizu.ac.ir*, Email:[b] *a.khalili@tabrizu.ac.ir*
[2]Cankaya University, Department of Mathematics and Computer Science, Ankara, Turkey
[3]Institute of Space Sciences, P.O. BOX, MG-23, RO 76900, Magurele-Bucharest, Romania
Email:[c] *dumitru@cankaya.edu.tr*

Compiled July 22, 2017

Abstract. In this manuscript, we extend the F^α-calculus by suggesting theorems analogous to the Green's and the Stokes' ones. Utilizing the F^α-calculus, the classical multipole moments are generalized to fractal distributions. In addition, the generalized model for the Bohr's energy loss involving heavy charged particles is given.

Key words: Bohr's energy loss, multipole moments, fractal distributions.

1. INTRODUCTION

Fractals can be observed in several physical phenomena [1]. To study fractals, mathematicians have developed several methods and techniques [2, 3]. Especially, the applications of fractals were studied by Mandelbrot [4]. Recently, a method analogous to the ordinary calculus has been developed on fractals; this calculus was called F^α-calculus [5–9]. F^α-calculus is a successful theory proposed to solve the nonlinearity problem of the theories of anomalous diffusion and of other physical phenomena [10–33].

The applicability and simplicity of F^α-calculus has motivated us to expand the application area of this calculus to the fractal physical systems. Firstly, using the F^α-calculus, we prove divergence, Green's, and Stokes' theorems for fractals. In addition, we define fractal forms and then we use F^α-calculus to expand the classical multipole moments for fractal distributions [34–36]. Finally, we use the master equation to the physical process of passing of fast heavy particle through matter using the local fractional derivative. We present the classical background to energy straggling phenomenon and explain the value of this study and highlight the shortcomings of the application of the classical theory energy straggling. When a beam of fast heavy charged particles passes through continuum matter the particles lose energy stochastically [37, 38]. Suppose that a beam of heavy charged particles with kinetic energy T passes through a thickness Δx of an absorber and $f(T,x)dT$ is the fraction of the particles with energy between T and $T+dT$ on a position x through the absorber. It is obvious that $f(T,x)$ satisfies the following master equation:

$$\frac{\partial f(T,x)}{\partial x} = -\int_0^\infty q(T,\epsilon)f(T,x)d\epsilon + \int_0^\infty q(T+\epsilon,\epsilon)f(T+\epsilon,x)d\epsilon, \tag{1}$$

where $q(T,\epsilon)$ is the probability that a particle will lose an amount of ϵ - $\epsilon + d\epsilon$ in traversing a distance Δx in the absorber. $f(T,x)$ is a Gaussian distribution for thick absorbers [37, 39], and for thin layers $f(T,x)$ has heavy tail [38, 40–43]. Classical studies of transporting of heavy charged particles deal with continuum materials and differentiable transition probability functions, $q(T,\epsilon)$. Consequently, there is a knowledge gap in the field of study of fractal materials and more general energy transition probabilities, for example, nowhere differentiable functions and fractal functions. To study more general energy transition probabilities, we expand the master equation with local fractional derivative.

This paper is divided into six Sections. The next Section is devoted to introduce the mathematical tools that we use in this manuscript. Section 3 is about the expansion of Green's, Stokes', and the divergence theorems and fractal multipole moments. In Sec. 4 the application of the local fractional derivative to calculate the straggling function in fractal structures is developed. The 5th Section discuses extension of classical Bohr's energy loss for fractals. Finally we present a summary of our conclusions.

2. BASIC TOOLS

In this Section, we give an introduction to the mathematical tools used in this paper.

2.1. A LOCAL FRACTIONAL DERIVATIVE

The local fractional α-order derivative of function f is defined as

$$D^\alpha f(y) = \lim_{x\to y} \frac{d^\alpha [f(x) - f(y)]}{[d(x-y)]^\alpha} \qquad 0 \le \alpha \le 1, \tag{2}$$

where the right hand side is the Riemann-Liouville fractional derivative which is defined by [7]

$$\frac{d^\alpha f(x)}{[d(x-a)]^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} dy \qquad 0 \le \alpha \le 1. \tag{3}$$

This kind of fractional derivative has many successful applications. For example, it is a local operator and its function on a constant quantity results zero [7]. It is also proved that there exits a quantitative connection between the box dimension of nowhere differentiable functions and the existence of the order of local fractional derivative [7]. Furthermore, the local fractional derivative appears as the coefficient of the power with fractional exponent i.e.

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)} \Delta^n + \frac{D^\alpha f(y)}{\Gamma(\alpha+1)} (\pm\Delta)^\alpha + R_\alpha(y,\Delta), \tag{4}$$

where $R_\alpha(y,\Delta)$ is the remainder and $\Delta = x - y$ [8].

2.2. FRACTAL CALCULUS ON A SUBSET OF R

This Section contains an introduction to F^α-calculus. We choose the important definitions and theorems of F^α-calculus, and for more details, we refer the readers to the references mentioned in the introduction Section. In this paper, we assume that F represents all fractal structures with dimension α. In addition, the notation of this calculus can be seen in [5, 6].

Definition 2.1 *The basis of F^α- calculus is on the integral staircase function, $S_F^\alpha(x)$, of order α for a fractal set F, which is given as follows:*

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,a,x), & \text{if } x \geq a; \\ -\gamma^\alpha(F,x,a), & \text{otherwise,} \end{cases} \tag{5}$$

where γ^α is the mass function of the fractal set F, all $\alpha, x, a \in \Re$ and $0 < \alpha \leq 1$ [6].

It is worth pointing that the two important properties of S_F^α, the continuity and monotonic increasing properties, are the essence of the definitions of $F^\alpha-$derivative and $F^\alpha-$integral on fractals.

Definition 2.2 *Let $F \subset R$, $f : R \to R$ and $x \in F$. A number l is said to be the limit of f through the points of F, or simply F-limit of f, as $y \to x$, for any ϵ there exists $\delta > 0$ such that*

$$y \in F, |y - x| < \delta \Rightarrow |f(y) - l| < \epsilon, \tag{6}$$

then it is denoted by

$$l = F - lim_{y \to x} f(y), \tag{7}$$

[6].

Now the definitions of F^α-derivative and F^α-integral of fractals are as follows:

Definition 2.3 *If F is an α-perfect set then the F^α-derivative of f at x is*

$$D_F^\alpha f(x) = \begin{cases} F - \lim_{y \to x} \frac{f(y) - f(x)}{S_F^\alpha(y) - S_F^\alpha(x)}, & x \in F; \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

[6].

Definition 2.4 *Let f be a fractal function. Let I be a closed interval. Then*

$$M[f,F,I] = \sup_{x\in F\cap I} f(x) \quad if F\cap I \neq \varnothing, \tag{9}$$
$$= 0 \quad otherwise,$$

and similarly

$$m[f,F,I] = \inf_{x\in F\cap I} f(x) \quad if F\cap I \neq \varnothing, \tag{10}$$
$$= 0 \quad otherwise,$$

[6].

Definition 2.5 *Let S_F^α be finite for $x \in [a,b]$. Let P be a subdivision of $[a,b]$ with points $x_0,...,x_n$. The upper F^α -sum and lower F^α-sum for the function f over the subdivision P are given, respectively, by*

$$U^\alpha[f,F,I] = \sum_{i=0}^{n-1} M[f,F,[x_i,x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)), \tag{11}$$

and

$$L^\alpha[f,F,I] = \sum_{i=0}^{n-1} m[f,F,[x_i,x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)), \tag{12}$$

[6].

Definition 2.6 *Let F be such that S_F^α is finite on $[a,b]$. For f, a function on F, the lower F^α-integral is given by*

$$\underline{\int_a^b} f(x) d_F^\alpha x = \sup_{P_{[a,b]}} L^\alpha[f,F,P], \tag{13}$$

the upper F^α-integral is given by

$$\overline{\int_a^b} f(x) d_F^\alpha x = \inf_{P_{[a,b]}} U^\alpha[f,F,P], \tag{14}$$

[6].

Definition 2.7 *Let f be a fractal set F, f is F^α-integrable on $[a,b]$ if*

$$\overline{\int_a^b} f(x) d_F^\alpha x = \underline{\int_a^b} f(x) d_F^\alpha x, \tag{15}$$

and it will be denoted by

$$\int_a^b f(x) d_F^\alpha x, \tag{16}$$

[6].

Theorem 2.1 *A function h is α-integrable over $[a,b]$ if and only if $g=\phi[h]$ (g is conjugate function of h) is Riemann integrable over $K=[S^{\alpha}(a),S^{\alpha}(b)]$*

$$\int_a^b h(x)d^{\alpha}x=\int_{S^{\alpha}(a)}^{S^{\alpha}(b)} g(u)du. \tag{17}$$

Theorem 2.2 *Let h be a function such that the image $g=\phi[h]$ of h is ordinary differential on K. Then*

$$D_F^{\alpha}h(x)=\frac{dg(t=S^{\alpha}(x))}{dt}, \tag{18}$$

for all $x\in F$ [6].

The stair function satisfies

$$ax^{\alpha}\leq S_F^{\alpha}(x)\leq bx^{\alpha}, \tag{19}$$

where a and b are constants [5, 6].

3. FRACTAL GREEN'S, STOKES', AND THE DIVERGENCE THEOREMS

In this Section, we extend fundamental theorems of classical calculus for a fractal set. We use these theorems in the next subsection. Firstly, let us define exterior derivative as follows [22];

$$d^{\alpha}=\sum_{i=1}^{n} d_F^{\alpha}x_i D_{F,x_i}^{\alpha}, \tag{20}$$

and, for $n=1$ it is given by

$$d^{\alpha}f=D_{F,x}^{\alpha}f(x)d_F^{\alpha}x, \tag{21}$$

where f is a function on the fractal set F. The integration of Eq. (21) is as follows;

$$\int_F d^{\alpha}f=\int_F D_F^{\alpha}f(x)d_F^{\alpha}x=f(b)-f(a). \tag{22}$$

For three dimensions we have

$$d^{\alpha}f=D_{F,x}^{\alpha}f d_F^{\alpha}x+D_{F,y}^{\alpha}f d_F^{\alpha}y+D_{F,z}^{\alpha}f d_F^{\alpha}z, \tag{23}$$

or in another notation

$$d^{\alpha}f=\nabla_F^{\alpha}f\cdot d_F^{\alpha}\mathbf{r}. \tag{24}$$

Let us calculate the fractal surface integral on a fractal cubic S with fractal boundaries

$$\int_S f_z d_F^{\alpha}x d_F^{\alpha}y+f_y d_F^{\alpha}x d_F^{\alpha}z+f_x d_F^{\alpha}z d_F^{\alpha}y=\int_S \mathbf{f}\cdot d_F^{\alpha}\mathbf{a}, \tag{25}$$

where $\mathbf{f} = (f_x, f_y, f_z)$ is a fractal vector filed on fractal space. Along z axis on fractal boundary surfaces of S_1 and S_2

$$\int_{S_1} f_z da_z + \int_{S_2} f_z da_z = \int_a^b \int_c^d f_z(x,y,e) d_F^\alpha y d_F^\alpha x - \int_a^b \int_c^d f_z(x,y,f) d_F^\alpha y d_F^\alpha x, \tag{26}$$

where a,b,c,d are the boundaries on which the fractal set is defined. By simplification

$$\int_a^b \int_c^d \left[f_z(x,y,e) - f_z(x,y,f) \right] d_F^\alpha y d_F^\alpha x, \tag{27}$$

and using

$$\int_e^f D_{F,z}^\alpha f_z(z) d_F^\alpha z = f(x,y,e) - f(x,y,f). \tag{28}$$

and repeating the above calculation on x axis and y axis, we obtain

$$\int_V \mathbf{D}_F^\alpha \cdot \mathbf{f}\, d^\alpha V = \int_S \mathbf{f} \cdot d_F^\alpha \mathbf{a}, \tag{29}$$

where V is the fractal volume of fractal surface S and it is called fractal version of divergence theorem. Now let us prove the Green's theorem

$$\oint_{\partial S} M d_F^\alpha x + N d_F^\alpha y = \int \int_S \left[D_{F,y}^\alpha N - D_{F,x}^\alpha M \right] d_F^\alpha y d_F^\alpha x, \tag{30}$$

where ∂S is the fractal boundary of fractal area S and M,N are the functions on fractal set. The righ thand side of Eq. (30) in a fractal square can be written as

$$\int \int -D_{F,y}^\alpha M d_F^\alpha y d_F^\alpha x = \int_a^b \left[M(x,c) - M(x,d) \right] d_F^\alpha x, \tag{31}$$

and the left hand side of the equation

$$\oint M d^\alpha x = \int_1 M d_F^\alpha x + \int_2 M d_F^\alpha x,, \tag{32}$$

$$= \int_a^b M(x,c) d_F^\alpha x + \int_b^a M(x,d) d_F^\alpha x,$$

$$= \int_a^b \left[M(x,c) - M(x,d) \right] d_F^\alpha x,$$

where the indices 1 and 2 are the line boundaries of square with intervals a,b at $y = c, y = d$. By using the same method we can prove

$$\oint N d_F^\alpha y = \int \int D_{F,y}^\alpha N d_F^\alpha y d_F^\alpha x. \tag{33}$$

Using the fractal Green's theorem, in the similar way, we can prove the Stokes' theorem,

$$\oint_c \mathbf{f}\cdot d^{\alpha}\mathbf{r} = \int \nabla_F^{\alpha}\times\mathbf{f}\cdot d^{\alpha}\mathbf{a}. \tag{34}$$

3.1. APPLICATION

With the new integral definition on fractals we propose the fractal integral Maxwell equations. The classical Maxwell equations describe electric and magnetic fields on continuous materials. So, they are not applicable on fractal materials. We suggest the classical Maxwell's equations and their fractal forms being as follows;

$$\epsilon_0 \oint_S \mathbf{E}\cdot d\mathbf{a} = \int_V \rho dV, \tag{35}$$

(Gauss' law),

$$\oint_{\partial S} \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{a}, \tag{36}$$

(Faraday's law),

$$\oint_S \mathbf{B}\cdot d\mathbf{a} = 0, \tag{37}$$

(Gauss' law of magnetic field),

$$\oint_{\partial S} \mathbf{B}\cdot d\mathbf{l} = \mu_0\int_S \mathbf{J}\cdot d\mathbf{a} + \epsilon_0\mu_0\frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d\mathbf{a}, \tag{38}$$

(Ampere's law),

$$\epsilon_0 \oint_S \mathbf{E}\cdot d^{\alpha}\mathbf{a} = \int_V \rho d^{\alpha}V, \tag{39}$$

(Fractal Gauss' law),

$$\oint_{\partial S} \mathbf{E}\cdot d^{\alpha}\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d^{\alpha}\mathbf{a}, \tag{40}$$

(Fractal Faraday's law),

$$\oint_S \mathbf{B}\cdot d^{\alpha}\mathbf{a} = 0, \tag{41}$$

(Fractal Gauss' law of magnetic field),

$$\oint_{\partial S} \mathbf{B}\cdot d^{\alpha}\mathbf{l} = \mu_0\int_S \mathbf{J}\cdot d^{\alpha}\mathbf{a} + \epsilon_0\mu_0\frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d^{\alpha}\mathbf{a}, \tag{42}$$

(Fractal Ampere's law).

Here **dl** is the vector along a line, **da** is the vector area, dV is the volume, S is the surface, ∂S is the boundary of surface. For classical and fractal electromagnetic equations we refer to [21, 22, 44].

The line integral of the magnetic field **B** along a fractal closed curve, **l**, is given by

$$\oint \mathbf{B}\cdot d^{\alpha}\mathbf{l} = \mu_0 I, \tag{43}$$

where I is given by

$$I = 2\pi \int J(r) r d^{\alpha} r. \tag{44}$$

Here $J(r)$ is the fractal cylindrically symmetric current density distribution and r is the radius of the coordinate. Then the left hand side of Eq. (43) becomes

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2\pi a B(a). \tag{45}$$

For $J(r) = J_0$, Eq. (44), Eq. (45), and Eq. (17), lead to

$$B(a) = \int_0^a J_0 S_F^{\alpha}(r) dS_F^{\alpha}(r), \tag{46}$$

and using Eq. (19) we conclude that

$$B(a) = J_0 \frac{1}{4a}(S_F^{\alpha}(a))^2 \sim a^{2\alpha-1}. \tag{47}$$

At this point we note that this result differs from the other fractal theories [35, 36].

3.2. ELECTRIC QUADRUPLE EXPANSION FOR FRACTAL CHARGE DISTRIBUTION

The quadrupole terms for potential is given by

$$V_2 = \frac{1}{4\pi\epsilon}\frac{1}{a^3}\int_F r^2 P_2(\cos\theta)\rho(r) d^{\alpha}V, \tag{48}$$

by replacing $P_2(\cos\theta)$ we have

$$= \frac{1}{4\pi\epsilon}\frac{1}{a^3}\int_F r^2 (\frac{3}{2}\cos^2\theta - \frac{1}{2})\rho(r) d^{\alpha}V, \tag{49}$$

where θ is the angle between **a** and **r** vectors

$$= \frac{1}{4\pi\epsilon}\frac{1}{2a^3}\int (\frac{3}{a^2}(\mathbf{a}\cdot r)^2 - r^2)\rho(\mathbf{r}) d^{\alpha}V, \tag{50}$$

and then

$$V_2 = \frac{1}{4\pi\epsilon}\frac{1}{2a^3}\sum_{k,l=1}^{3}\frac{X_k X_l}{a^2} Q_{kl}, \tag{51}$$

where X_k are Cartesian's coordinate of **a**, and the electric quadruple is defined by

$$Q_{lk}^{\alpha} = \int_F [3x_l x_k - r^2\delta_{kl}]\rho(\mathbf{r}) d^{\alpha}V, \tag{52}$$

where x_k are coordinates of the vector **r**. For simplicity, we consider the more general form of Q^α

$$Q^\alpha(\alpha,\beta,\gamma) = \int_F [\alpha x^2 + \beta y^2 + \gamma z^2]\rho(r) d^\alpha V, \tag{53}$$

then for fractal parallelepiped with volume $0 \le x \le A$, $0 \le x \le B$, $0 \le x \le C$ we have

$$Q^\alpha = \frac{1}{3} S_F^\alpha(A) S_F^\alpha(B) S_F^\alpha(C)[\alpha S_F^\alpha(A)^2 + \beta S_F^\alpha(B)^2 + \gamma S_F^\alpha(C)^2], \tag{54}$$

where $Q^\alpha = \rho_0 S^\alpha(A) S^\alpha(B) S^\alpha(C)$ is the electric charge of the fractal distribution.

4. EXTENSION OF THE MASTER EQUATION

If the maximum energy loss in any single collision is small, then the right hand side of Eq. (1) can be expanded as

$$\begin{aligned} -\int_0^\infty q(T,\epsilon) f(T,x) d\epsilon + \int_0^\infty q(T+\epsilon,\epsilon) f(T+\epsilon,x) d\epsilon = \\ \sum_{k=1}^{\infty} \frac{\partial^k}{\partial T^k}\big[N_k(T) f(T,x)\big], \end{aligned} \tag{55}$$

where N_k is defined as

$$N_k(T) = \int_0^\infty d\epsilon \epsilon^k q(T,\epsilon). \tag{56}$$

But if the energy transitional probability function is a non differential function this expansion would be invalid, so, to generalize this equation we use the local fractional Taylor expansion. Then, Eq. (1) can be written as

$$\frac{\partial f}{\partial x} = \sum_{k=1}^{\infty} \frac{\partial^k}{\partial T^k}\Big[N_k(T) f(T,x)\Big] + D_T^\alpha \Big[\frac{N_\alpha(T)}{\Gamma(\alpha+1)} f(T,x)\Big]. \tag{57}$$

where

$$N_\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\alpha q(T,\epsilon)\epsilon^\alpha d\epsilon. \tag{58}$$

If $0 < \alpha < 1$, then Eq. (57) becomes

$$\frac{\partial f(T,x)}{\partial x} \approx D_T^\alpha\big[N_\alpha(T) f(T,x)\big]. \tag{59}$$

Using Eq. (58), we propose a fractal model for the thick absorbers as

$$q(T,\epsilon) = \frac{k_R}{T\epsilon^{1+\alpha}}, \tag{60}$$

where k_R, β are constants that depend on the incident particle properties. For $\alpha = 1$ we lead to the classical transition probability $q(T,\epsilon) = \frac{k_R}{T\epsilon^2}$ [37]. In the following section we consider the space to be fractal and calculate the transition probability numerically.

5. EXTENSION OF THE CLASSICAL BOHR'S ENERGY LOSS CALCULATIONS

In this Section, we give a physical fractal model to the transition of energy. Consider a heavy particle with a charge ze, mass M and velocity v passing through a fractal medium. Suppose that there is a free and rest electron at some distance b from the particle path. After the collision we assume the heavy particle to be undeviated from its trajectory because of its larger mass $(M \gg m_e)$. We find the momentum impulse electron receives from the collision with the heavy particle;

$$J = \int F dt = e \int E_{\perp} \frac{d^{\alpha}x}{v}, \tag{61}$$

where $E_{\perp}$ is the vertical component of electric field to the particle path. Using the extension of electromagnetic laws we have

$$\int E_{\perp} d^{\alpha}x = \frac{2ze}{b}, \tag{62}$$

so that

$$J = \frac{2ze^2}{bv}, \tag{63}$$

and the energy lost by an electron will be

$$\Delta T(b) = \frac{J^2}{2m_e} = \frac{2z^2e^4}{m_e v^2 b^2}. \tag{64}$$

If N_e is the density of electrons, the energy lost by all the electrons at dV is

$$dT(b) = \Delta T(b) N_e dV = \frac{2z^2e^4}{m_e v^2 b^2} N_e \frac{d^{\alpha}b}{b} d^{\alpha}x, \tag{65}$$

where we have assumed that the fractal dimension of radius space is the same as x space. Then Eq. (65) becomes

$$D_x^{\alpha} T = \frac{2z^2e^4}{m_e v^2 b^2} N_e \int \frac{d^{\alpha}b}{b}. \tag{66}$$

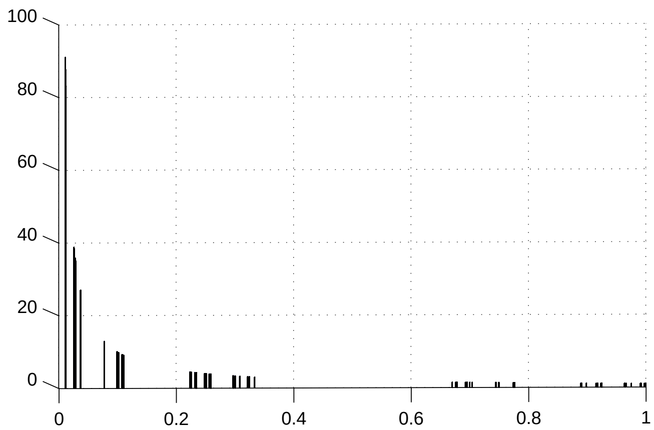

Fig. 1 – Fractal space is a Cantor set: $\frac{1}{b}\chi_C(b)$.

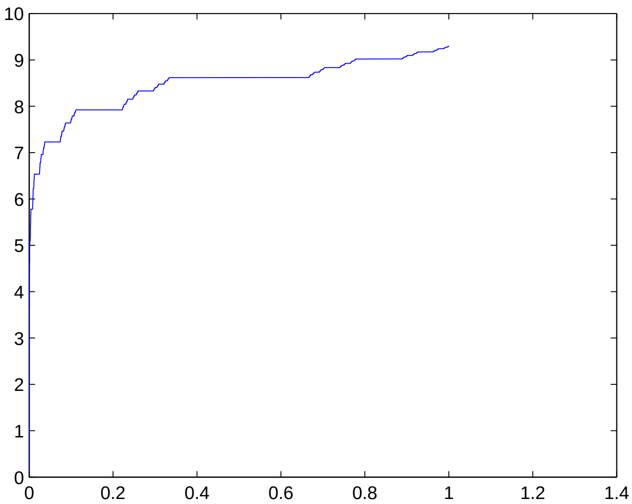

Fig. 2 – The numerical calculation of the integral $\int \frac{d^\alpha b}{b}\chi_C(b)$.

If we integrate Eq. (66) over Cantor space, we obtain the results shown in Figs. 1 and 2.

6. CONCLUSION

The Green's and the Stokes' theorems were generalized for functions with the fractal support. The F^{α}-calculus was used to calculate the multipole moments of the fractal charge distributions. In addition, we have used the Taylor expansion involving F^{α}-derivatives to treat the more general energy transition probability functions. We have also proposed a new model for the energy losing of particles through matter, which is useful in the case of non-differentiable functions. The advantages of the F^{α}-calculus is that can be applied for every fractal sets and fractal curves.

REFERENCES

1. A. Bunde and S. Havlin (Eds.), *Fractals in science*, Springer, 1995.
2. K. Falconer, *Fractal geometry: Mathematical foundations and applications*, John Wiley & Sons, 2004.
3. K. Falconer, *Techniques in fractal geometry*, John Wiley & Sons, 1997.
4. B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman and Company, 1977.
5. A. Parvate, S. Satin, and A. D. Gangal, Fractals **19**, 15–27 (2011).
6. A. Parvate and A. D. Gangal, Fractals **17**, 53–81 (2009).
7. K. M. Kolwankar and A. D. Gangal, Phys. Rev. Lett. **80**, 214 (1998).
8. K. M. Kolwankar and A. D. Gangal, Chaos **6**, 505–513 (1996).
9. K. M. Kolwankar and A. D. Gangal, Pramana **48**, 49–68 (1997).
10. J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127–293 (1990).
11. R. Metzler, E. Barkai, and J. Klafter, Phys. Rev. Lett. **82**, 3563 (1999).
12. R. Klages, G. Radons, and I. M. Sokolov, (Eds). *Anomalous transport: foundations and applications*, John Wiley & Sons, 2008.
13. R. Metzler, E. Barkai, and J. Klafter, Physica A **266**, 343–350 (1999).
14. R. Metzler, W. G. Glockle, and T. F. Nonnenmacher, Physica A **211**, 13–24 (1994).
15. R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, and P. Paradisi, Chem. Phys. **284**, 521–544 (2002).
16. I. Gottlieb, P. Nica, and M. Agop, Rom. Rep. Phys. **60**, 443–451 (2008).
17. B. O'Shaughnessy and I. Procaccia, Phys. Rev. Lett. **54**, 455 (1985).
18. M. Agop, V. Enache, and M. Buzdugan, Rom. J. Phys. **53**, 23–28 (2008).
19. X.-J. Yang, D. Baleanu, and W.-P. Zhong, Proc. Romanian Acad. A **14**, 127–133 (2013).
20. E. M. Anitas, Rom. J. Phys. **60**, 647–652 (2015).
21. A. K. Golmankhaneh and D. Baleanu, Rom. Rep. Phys. **69**, 109 (2017).
22. A. K. Golmankhaneh, A. K. Golmankhaneh, and D. Baleanu, Centr. Eur. J. Phys. **11**, 863–867 (2013).
23. A. K. Golmankhaneh, V. Fazlollahi, and D. Baleanu, Rom. Rep. Phys. **65**, 84–93 (2013).
24. C. Stan, M. Balasoiu, A. I. Ivankov, and C. P. Cristescu, Rom. Rep. Phys. **68**, 270–277 (2016).
25. K. Nouri, S. Elahi-Mehr, and L. Torkzadeh, Rom. Rep. Phys. **68**, 503–514 (2016).
26. A. Y. Cherny, E. M. Anitas, V. A. Osipov, and A. I. Kuklin, Rom. J. Phys. **60**, 658–663 (2015).
27. A. H. Bhrawy, Proc. Romanian Acad. A **17**, 39–47 (2016).
28. Y. Zhang, D. Baleanu, and X. J. Yang, Proc. Romanian Acad. A **17**, 230–236 (2016).

29. M. A. Abdelkawy, M. A. Zaky, A. H. Bhrawy, and D. Baleanu, Rom. Rep. Phys. **67**, 773–791 (2015).
30. W. M. Abd-Elhameed, and Y. H. Youssri, Rom. J. Phys. **61**, 795–813 (2016).
31. A. Agila, D. Baleanu, R. Eid, and B. Irfanoglu, Rom. J. Phys. **61**, 350–359 (2016).
32. D. Kumar, J. Singh, and D. Baleanu, Rom. Rep. Phys. **69**, 103 (2017).
33. X. J. Yang, F. Gao, and H. M. Srivastava, Rom. Rep. Phys. **69**, 113 (2017).
34. V. E. Tarasov, Mod. Phys. Lett. B. **19**, 1107–1118 (2005).
35. V. E. Tarasov, Mod. Phys. Lett. A **21**, 1587–1600 (2006).
36. V. E. Tarasov, Ann. Phys. **323**, 2756–2778 (2008).
37. M. G. Payne, Phys. Rev. **185**, 611 (1969).
38. W. R. Leo, *Techniques for Nuclear and Particle Phyisics Experiments*, Springer-Verlag, Berlin Heidelberg, 1994.
39. U. Fano, Ann. Rev. Nucl. Sci. **13**, 1 (1963).
40. P. V. Vavilov, Zh. Eksp. Teor. Fiz. **32**, 920 (1957).
41. N. Tsoulfanidis and L. Sheldon, *Measurement and detection of radiation*, CRC Press, 2011.
42. G. F. Knoll, Radiation detection and measurement, Wiley, 2011.
43. L. D. Landau, J. Phys. **8**, 201 (1944).
44. J. R. Reitz, J. M. Frederick, and W. C. Robert, *Foundations of electromagnetic theory*, Addison-Wesley Publishing Company, 2008.