$V(x^i)$ FUNCTIONS INDUCED BY ALGEBRAICALLY SPECIAL VACUUM SPACES

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This paper describes the problem of finding potential functions admitted by Klein-Gordon equations, via a method based on a geometric selection rule. We exemplify the method by applying it to geometries that admit simply-transitive group of motions, in particular, Petrov type II, III, N and D. We derive the functional forms of the potentials under certain restrictions involving subgroups and linear combinations of the homothety group. Due to the volume of results and for utilizing space economically, the results are presented in the form of tables.

Key words: Petrov spaces, Klein-Gordon equation, Lie symmetries.

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1. INTRODUCTION

A special class of Petrov spaces that are of considerable interest in contemporary physics are the algebraically special vacuum solutions of Einstein's equations. These are the spaces for which the homothetic vectors (HVs) acts simply transitively and are described together with their homothetic algebras in [1]. Specifically, these spaces are known as Petrov type II, III, N and D.

We investigate Klein-Gordon equations in a general space, generated by the metric g,

$$\frac{1}{\sqrt{|-g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|-g|}g^{ij}\frac{\partial u}{\partial x^j}\right)+V(x^i)u=0, \tag{1}$$

that is derived from the Lagrangian

$$L\left(x^i,u,u_i\right)=\frac{1}{2}\sqrt{g}\,g^{ij}u_iu_j-\frac{1}{2}\sqrt{g}\,V(x^i)\,u^2. \tag{2}$$

In a Riemannian space of dimension n, the elements of the conformal algebra of the metric provide the Lie point symmetry vectors

$$X=\xi^i(x^k)\partial_i+\left(\frac{2-n}{n}\psi(x^k)u+a_0u+b(x^k)\right)\partial_u, \quad n>2$$

of the Klein-Gordon equation, where ξ^k is a conformal Killing vector (CKV) with conformal factor $\psi(x^k), b(x^k)$ is a solution of (1) and additionally, the following geometric selection rule concerning the potential V is satisfied [2]

$$\xi^k V_k + 2\psi V - \frac{2-n}{2}\triangle\psi = 0. \tag{3}$$

Indeed, several works exist in the recent literature which delve into potentials. For example, the diagonal Bianchi I spacetime was split into the conformally and non-conformally flat spaces and were analyzed in terms of potential functions in [3], while the classification of the space-time homogenous Gödel-type metrics lead to interesting potentials in [4]. For a concise discussion of the extensions and applications related to these potentials, see [5]. Our purpose is to derive the potential functions of the Klein-Gordon equation under each of the algebraically special vacuum Petrov spaces.

At this point it may be appropriate to mention the reason why we are interested in the homothety group of the Petrov spaces. A classification of subgroups of Lie symmetry groups of the Klein-Gordon differential equations is an essential part in the study of the potentials. In the work to come, we require certain group features of the Petrov spaces under consideration, such as the homothety group and its real subgroups. Primarily, we shall use the subgroup structure to obtain potentials for an underlying Klein-Gordon equation.

With this in mind, in a previous work [6], we showed that in n-dimensional space, we need all real $(n-1)$-dimensional subalgebras to obtain potentials of the Klein-Gordon equation that is then reducible by Lie invariant methods. Hence, we may find exact solutions easily and in addition, this requirement also ensures that only non constant potential functions are derived. Of course , one may also look at n-dimensional subgroups but in the present paper, most of the groups itself are of dimension n. Moreover, extra results are obtainable if we include subalgebras containing linear combinations of the vector fields [6]. Here, we omit such considerations for simplicity. Finally, we remark that the Klein-Gordon equation is a linear equation which implies that it will always admit the Lie point symmetry called the linear symmetry $P_{(u)} = u\partial_u$ and the infinite dimensional abelian subalgebra of solutions $P_{(\infty)} = F(x^i)\partial_u$, where $F(x^i)$ is a solution of the Klein-Gordon equation.

A discussion of symmetry based methods is given in the Appendix and we refer the interested reader for further details of the relevant equations and formulae to, inter alia, [7, 8]. The plan of the paper follows. Section 2, 3 and 4 contains our main focus whereby we recap the nomenclature of the homothetic algebra of the Petrov type II, III, N and D geometries. In particular, we define the potentials admitted by the Klein-Gordon equation based on a geometric selection criterion. Finally in Section 5 we draw our conclusions.

2. THE PETROV TYPE II SPACE

The Petrov type II space below is a special case of van Stockum's stationary axisymmetric metric [1], with line element

$$ds^2 = \mu^{-\frac{1}{2}}(d\mu^2 + dz^2) - 2\mu dx dy + \mu \ln\mu dy^2, \tag{4}$$

that admits a 4-dimensional homothetic algebra, which we label $\mathcal{A}_1$, viz.

$$\begin{gathered} W^1 = \partial_x, \quad W^2 = \partial_y, \quad W^3 = \partial_z, \\ W^4 = (x+2y)\partial_x + y\partial_y + 4z\partial_z + 4\mu\partial_\mu. \end{gathered} \tag{5}$$

For the sake of completeness, we mention the respective Euler Lagrangian equations of the geodesic Lagrangian and the Noetherian first integrals. The space has the geodesic Lagrangian

$$L_{II} = \frac{1}{2}\mu^{-\frac{1}{2}}(\dot{\mu}^2 + \dot{z}^2) - \mu\dot{x}\dot{y} + \frac{1}{2}\mu\ln\mu\dot{y}^2.$$

The Lagrangian gives rise to the field equations which are the Euler-Lagrange equations with respect to the variables $\{x, y, \mu, z\}$, where dot refers to a derivative with respect to the arclength parameter s, videlicet

$$(\ddot{x}, \ddot{y}, \ddot{\mu}, \ddot{z}) = \left(-\frac{\dot{\mu}(\dot{x}-\dot{y})}{\mu}, -\frac{\dot{\mu}\dot{y}}{\mu}, \frac{1}{4}\frac{2\ln(\mu)\dot{y}^2\mu^{3/2} - 4\dot{x}\dot{y}\mu^{3/2} + 2\dot{y}^2\mu^{3/2} + \dot{\mu}^2 - \dot{z}^2}{\mu}, \frac{1}{2}\frac{\dot{z}\dot{\mu}}{\mu} \right).$$

Conservation laws are admitted by application of the celebrated Noether's Theorem [9]:

$$\begin{aligned} I_{W^1} &= \dot{y}\mu, \\ I_{W^2} &= -\mu(\ln(\mu)\dot{y} - \dot{x}), \\ I_{W^3} &= -\frac{\dot{z}}{\sqrt{\mu}}, \\ I_{W^4} &= -\frac{1}{3}\frac{\mu^{3/2}\ln(\mu)y\dot{y} - \dot{y}\mu^{3/2}x - \mu^{3/2}\dot{x}y - 2\dot{y}\mu^{3/2}y + \dot{\mu}\mu + 4\dot{z}z}{\sqrt{\mu}}. \end{aligned} \tag{6}$$

Note that the Klein-Gordon equation (1) admits the KVs and the HV of a space [10,11], which also coincide with the Noether symmetries of the geodesic Lagrangian with a constant gauge function [11]. Hence, for every Lie point symmetry of the Klein-Gordon equation, a corresponding Noetherian conservation law exists for the classical field equations.

The Lie Brackets of the symmetry vectors are shown in Table 1.

The Lagrangian

$$\mathcal{L}_{II} = \frac{u_x u_y}{\sqrt{\mu}} - \frac{1}{2}\mu u_\mu^2 + \frac{1}{2}\frac{\ln(\mu)u_x{}^2}{\sqrt{\mu}} - \frac{1}{2}\mu u_z{}^2 + \frac{1}{2}\sqrt{\mu}V(x,y,\mu,z)u^2$$

Table 1.

Lie Brackets of $\mathcal{A}_1$

$[W^i, W^j]$	W^1	W^2	W^3	W^4
W^1	0	0	0	$\frac{W^1}{3}$
W^2	0	0	0	$\frac{2W^1}{3}+\frac{X^2}{3}$
W^3	0	0	0	$\frac{4W^3}{3}$
W^4	$-\frac{W^1}{3}$	$-\frac{2W^1}{3}-\frac{W^2}{3}$	$-\frac{4W^3}{3}$	0

generates the Klein-Gordon equation of the Petrov type II space,

$$-\mu^{-3/2}\left(u_{xx}\ln(\mu)\sqrt{\mu}-u_{zz}\mu^2-\mu u_\mu-u_{\mu\mu}\mu^2+2\sqrt{\mu}u_{xy}\right)+V(x,y,\mu,z)u=0. \tag{7}$$

The selection rule Eq. (3) can be utilized in several ways. To begin with, we apply each symmetry vector of the homothetic algebra called $\mathcal{A}_i$, to (3) with $n=4$ and find the potential functions $V(x^i)$ for which it is satisfied. Next, as explained above, we take 3-dimensional real subgroups of each $\mathcal{A}_i$ coupled with Eq. (3) to determine other functional forms of the potential function. Last but not least, linear combinations of the elements of $\mathcal{A}_i$ also yield interesting potential functions. Obviously, many linear combinations exist and so we need to limit ourselves to selected linear combinations. In particular, we restrict ourselves to pairs of linear combinations.

The solution of the selection rule involves some cumbersome and sophisticated calculations and for the sake of brevity, we only list the potentials without the corresponding conditions.

Within Table 2 we define two functions involving LambertW (LW) functions, that is

$$l_1=LW\left(\frac{1}{2}xy^{-1}\exp(x/2y)\right),$$

and

$$l_2=LW\left(\frac{1}{2}(bx+3a)(by)^{-1}\exp(x/2y)\exp(3a/2by)\right).$$

3. THE PETROV TYPE III SPACE

The metric classified as the Petrov type III space is a singular type III Robinson-Trautman, expressed as [1]

$$ds^2=\frac{3}{2}xd\mu^2+2d\mu d\nu+\frac{\nu^2}{x^3}(dx^2+dy^2), \tag{8}$$

Table 2.

The Homothetic Algebra $\mathcal{A}_1$ and Potential Functions

Potential Function $V(x,y,\mu,z)$	**Lie Symmetry or Algebra**	**Noether Symmetry or Algebra**
$V(x,y,\mu,z)$	$P_{(u)}$	$\times$
$V(y,\mu,z)$	W^1	$\surd$
$V(x,\mu,z)$	W^2	$\surd$
$V(x,y,\mu)$	W^3	$\surd$
$e^{-6l_1}V\left(\frac{-2y\ln(y)-x}{y},\frac{16\mu y^4l_1^4}{x^4\exp(x/y)^2},\frac{16zy^4l_1^4}{x^4\exp(x/y)^2}\right)$	W^4	$\surd$
$V(\mu)$	$\langle W^1,W^2,W^3\rangle$	$\surd$
$\mu^{-3/2}V\left(\frac{z}{\mu}\right)$	$\langle W^1,W^2,W^4\rangle$	$\surd$
$y^{-6}V\left(\frac{\mu}{y^4}\right)$	$\langle W^1,W^3,W^4\rangle$	$\surd$
$V\left(y-\frac{bx}{a},\mu,z\right)$	aW^1+bW^2	$\surd$
$V\left(y,\mu,z-\frac{bx}{a}\right)$	aW^1+bW^3	$\surd$
$e^{-6l_2}V\left(\frac{-2y\ln(y)b-bx-3a}{yb},\frac{16\mu b^4y^4l_2^4}{x^4\exp(x/y)^2},\frac{16zb^4y^4l_2^4}{x^4\exp(x/y)^2}\right)$	aW^1+bW^4	$\surd$
$V\left(x,\mu,z-\frac{by}{a},\mu,z\right)$	aW^2+bW^3	$\surd$
$e^{-6l_1}V\left(\frac{-2y\ln(y)-x}{yb},\frac{16\mu y^4l_1^4}{x^4\exp(x/y)^2},\frac{4(4bz+3a)y^4l_1^4}{x^4\exp(x/y)^2}\right)$	aW^3+bW^4	$\surd$

and the geodesic Lagrangian is

$$L_{III}=\frac{3}{4}x\dot{\mu}^2+\dot{\mu}\dot{\nu}+\frac{1}{2}\frac{\nu^2}{x^3}(\dot{x}^2+\dot{y}^2),$$

which yields the Euler-Lagrange equations

$$(\ddot{x},\ddot{y},\ddot{\mu},\ddot{\nu})=\left(\frac{1}{4}\,\frac{3\,\dot{\mu}^2x^4+6\,\nu^2\dot{x}^2-6\,\nu^2\dot{y}^2-8\,\nu\dot{\nu}x\dot{x}}{\nu^2x},\frac{\dot{y}\,(3\,\nu\dot{x}-2\,\dot{\nu}x)}{\nu x},\frac{\nu\left(\dot{x}^2+\dot{y}^2\right)}{x^3},-\frac{3}{2}\,\frac{\dot{\mu}x^2\dot{x}+\nu\dot{x}^2+\nu\dot{y}^2}{x^2}\right).$$

This space admits a 4-dimensional homothetic algebra $\mathcal{A}_2$, viz.

$$X^1=\partial_y,\quad X^2=\partial_\mu,\quad X^3=\nu\partial_\nu-\mu\partial_\mu+2x\partial_x+2y\partial_y,\quad X^4=\nu\partial_\nu+\mu\partial_\mu. \tag{9}$$

Each X^i is a Noether symmetry and corresponds to the first integrals

$$\begin{aligned} I_{X^1} = & -\frac{\nu^2 \dot{y}}{x^3}, \\ I_{X^2} = & -\tfrac{3}{2}\dot{\mu}x - \dot{\nu}, \\ I_{X^3} = & \frac{1}{2}\,\frac{3\,\mu\dot{\mu}x^4 + 2\,\mu\dot{\nu}x^3 - 2\,\dot{\mu}\nu x^3 - 4\,\nu^2\dot{x}x - 4\,\nu^2\dot{y}y}{x^3}, \\ I_{X^4} = & -\dot{\mu}\nu - \tfrac{3}{2}\,\mu\dot{\mu}x - \mu\dot{\nu}. \end{aligned} \tag{10}$$

The Lie Brackets of the symmetry vectors are shown in Table 3. The Lagrangian

Table 3.

Commutator relations of $\mathcal{A}_2$

$[X^i, X^j]$	X^1	X^2	X^3	X^4
X^1	0	0	2 X^1	0
X^2	0	0	-X^2	X^2
X^3	-2 X^1	X^2	0	0
X^4	0	-X^2	0	0

$$\mathcal{L}_{III} = -\frac{\nu^2 u_\mu u_\nu}{x^3} + \frac{3}{4}\,\frac{\nu^2 {u_\nu}^2}{x^2} - \frac{1}{2}\,{u_x}^2 - \frac{1}{2}\,{u_y}^2 + \frac{1}{2}\,\frac{\nu^2 V(x,y,\mu,\nu)\,u^2}{x^3}$$

generates the Klein-Gordon equation of the Petrov type III space,

$$-\frac{1}{2\nu^2}\left(3\,u_{\nu\nu}x\nu^2 - 2\,u_{yy}x^3 - 2\,u_{xx}x^3 - 4\,u_{\nu\mu}\nu^2 + 6\,\nu u_\nu x - 4\,\nu u_\mu\right) + V(x,y,\mu,\nu)\,u = 0. \tag{11}$$

As before, the construction of the functional forms of the potentials is induced by the point symmetry group of the Klein-Gordon equation. Hence the steps outlined in the previous section ultimately lead to the results of Table 4.

4. THE PETROV TYPE N SPACE

The metric of the Petrov type N space is given by

$$ds^2 = dx^2 + dy^2 + 2d\mu d\nu + 2\ln(x^2 + y^2)d\mu^2, \tag{12}$$

which is a p.p-wave. A symmetry classification of wave equations in several p.p-wave spaces of interest has appeared in [12]. This metric generates the following homothetic algebra, labeled $\mathcal{A}_3$

$$\begin{gathered} Y^1 = \partial_\mu, \quad Y^2 = \partial_\nu, \quad Y^3 = y\partial_x - x\partial_y, \\ Y^4 = x\partial_x + y\partial_y + \mu\partial_\mu + (\nu - 2\mu)\partial_\nu, \end{gathered} \tag{13}$$

Table 4.

The Algebra $\mathcal{A}_2$ and Potential Functions

Potential Function $V(x,y,\mu,\nu)$	**Lie Symmetry or Algebra**	**Noether Symmetry or Algebra**
$V(x,y,\mu,\nu)$	$P_{(u)}$	$\times$
$V(x,\mu,\nu)$	X^1	$\surd$
$V(x,y,\nu)$	X^2	$\surd$
$V\left(\frac{y}{x},\mu\sqrt{x},\frac{\nu}{\sqrt{x}}\right)$	X^3	$\surd$
$\frac{1}{\mu^2}V\left(x,y,\frac{\nu}{\mu}\right)$	X^4	$\surd$
$V\left(\frac{\nu}{\sqrt{x}}\right)$	$\langle X^1,X^2,X^3\rangle$	$\surd$
$\nu^{-2}V(x)$	$\langle X^1,X^2,X^4\rangle$	$\surd$
$x^{-1}\mu^{-2}V\left(\frac{\nu}{\mu x}\right)$	$\langle X^1,X^3,X^4\rangle$	$\surd$
$x\nu^{-2}V\left(\frac{y}{x}\right)$	$\langle X^2,X^3,X^4\rangle$	$\surd$
$V\left(x,\mu-\frac{by}{a},\nu\right)$	aX^1+bX^2	$\surd$
$V\left(\frac{1}{2}\frac{2by+a}{xb},\mu\sqrt{x},\frac{\nu}{\sqrt{x}}\right)$	aX^1+bX^3	$\surd$
$\exp(-2by/a)V(x,\mu\exp(-by/a),\nu\exp(-by/a))$	aX^1+bX^4	$\surd$
$V\left(\frac{y}{x},\frac{\sqrt{x}(b\mu-a)}{b},\frac{\nu}{\sqrt{x}}\right)$	aX^2+bX^3	$\surd$
$(b\mu+a)^{-2}V\left(x,y,\frac{\nu}{b\mu+a}\right)$	aX^2+bX^4	$\surd$
$(x)^{-b/a}V\left(\frac{y}{x},\mu x^{\frac{a-b}{2a}},\nu x^{-\frac{a+b}{2a}}\right)$	aX^3+bX^4	$\surd$

and whose commutators are displayed in table 5.

Table 5.

Lie Brackets for $\mathcal{A}_3$

$[Y^i,Y^j]$	Y^1	Y^2	Y^3	Y^4
Y^1	0	0	0	Y^1-2Y^2
Y^2	0	0	0	Y^2
Y^3	0	0	0	0
Y^4	$-Y^1+2Y^2$	-Y^2	0	0

The geodesic Lagrangian of the space is expressed as

$$L_N=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+\dot{\mu}\dot{\nu}+\ln(x^2+y^2)\dot{\mu}^2,$$

and the resultant field equations or Euler-Lagrange equations are

$$(\ddot{x},\ddot{y},\ddot{\mu},\ddot{\nu}) = \left(\frac{x\dot{\mu}^2}{x^2+y^2},\frac{y\dot{\mu}^2}{x^2+y^2},0,-2\,\frac{\dot{\mu}\,(x\dot{x}+y\dot{y})}{x^2+y^2}\right).$$

Each Y^i corresponds to the first integral:

$$\begin{array}{rc} I_{Y^1} = & -\dot{\nu}-\ln\left(x^2+y^2\right)\dot{\mu}, \\ I_{Y^2} = & -\dot{\mu}, \\ I_{Y^3} = & \dot{y}x-\dot{x}y, \\ I_{Y^4} = & \dot{\mu}\,(-v+2\,\mu)+\left(-\dot{\nu}-\ln\left(x^2+y^2\right)\dot{\mu}\right)\mu-\dot{y}y-\dot{x}x. \end{array} \tag{14}$$

The Lagrangian

$$\mathcal{L}_N = -u_\nu u_\mu - \frac{1}{2}{u_x}^2 - \frac{1}{2}{u_y}^2 + \ln\left(x^2+y^2\right){u_\nu}^2 + \frac{1}{2}V\,(x,y,\mu,\nu)\,u^2$$

generates the Klein-Gordon equation of the Petrov type N space,

$$-2u_{\nu\nu}\ln\left(x^2+y^2\right)+2\,u_{\nu\mu}+u_{xx}+u_{yy}+V\,(x,y,\mu,\nu)\,u = 0. \tag{15}$$

Similarly, the symmetry vectors, their linear combinations and subgroups provide the potentials of Table 6.

5. THE PETROV TYPE D SPACE

The line element corresponding to Petrov type D space is equivalent to a Kasner metric [1]:

$$ds^2 = -dx^2 + x^{-\frac{2}{3}}dy^2 - x^{\frac{4}{3}}(d\mu^2+dz^2). \tag{16}$$

The geodesic Lagrangian

$$L_D = \frac{1}{2}\left(-\dot{x}^2 + x^{-\frac{2}{3}}\dot{y}^2 - x^{\frac{4}{3}}(\dot{\mu}^2+\dot{z}^2)\right),$$

provides the field equations

$$(\ddot{x},\ddot{y},\ddot{\mu},\ddot{z}) = \left(\frac{1}{3}\,\frac{2\,\dot{\mu}^2x^2+2\,x^2\dot{z}^2+\dot{y}^2}{x^{5/3}},\frac{2}{3}\,\frac{\dot{y}\dot{x}}{x},-\frac{4}{3}\,\frac{\dot{\mu}\dot{x}}{x},-\frac{4}{3}\,\frac{\dot{z}\dot{x}}{x}\right).$$

This space admits a homothetic algebra $\mathcal{A}_4$ (Lie Brackets are presented in 7), of

$$\begin{array}{c} Z^1 = \partial_y, \quad Z^2 = \partial_\mu, \quad Z^3 = \partial_z, \\ Z^4 = z\partial_\mu - \mu\partial_z \\ Z^5 = 3x\partial_x + 4y\partial_y + z\partial_z + \mu\partial_\mu. \end{array} \tag{17}$$

Table 6.

The Algebra $\mathcal{A}_3$ and Potential Functions

Potential Function $V(x,y,\mu,\nu)$	**Lie Symmetry or Algebra**	**Noether Symmetry or Algebra**
$V(x,y,\mu,\nu)$	$P_{(u)}$	$\times$
$V(x,y,\nu)$	Y^1	$\surd$
$V(x,y,\mu)$	Y^2	$\surd$
$V\left(x^2+y^2,\mu,\nu\right)$	Y^3	$\surd$
$\frac{1}{x^2}V\left(\frac{y}{x},\frac{\mu}{x},\frac{2\ln(x)\mu+\nu}{x}\right)$	Y^4	$\surd$
$V\left(x^2+y^2\right)$	$\langle Y^1,Y^2,Y^3\rangle$	$\surd$
$x^{-2}V\left(\frac{y}{x}\right)$	$\langle Y^1,Y^2,Y^4\rangle$	$\surd$
$\frac{1}{x^2+y^2}V\left(\frac{\mu}{\sqrt{x^2+y^2}}\right)$	$\langle Y^2,Y^3,Y^4\rangle$	$\surd$
$V\left(x,y,\nu-\frac{b\mu}{a},\nu\right)$	aY^1+bY^2	$\surd$
$V\left(x^2+y^2,\mu-a/b\arctan(x/y),\nu\right)$	aY^1+bY^3	$\surd$
$x^{-2}V\left(\frac{y}{x},\frac{b\mu+a}{xb},\frac{2(b\mu+a)\ln(x)\mu+b\nu+2a}{xb}\right)$	aY^1+bY^4	$\surd$
$V\left(x^2+y^2,\mu,\frac{b\nu-a\arctan(x/y)}{b}\right)$	aY^2+bY^3	$\surd$
$x^{-2}V\left(\frac{y}{x},\frac{\mu}{x},\frac{2b\mu\ln(x)+\nu b+a}{bx}\right)$	aY^2+bY^4	$\surd$

The corresponding conserved quantities are

$$\begin{aligned}
I_{Z^1} &= -\frac{\dot{y}}{x^{2/3}},\\
I_{Z^2} &= \dot{\mu}x^{4/3},\\
I_{Z^3} &= x^{4/3}\dot{z},\\
I_{Z^4} &= -x^{4/3}\dot{z}\mu+\dot{\mu}x^{4/3}z,\\
I_{Z^5} &= \frac{1}{3}\,\frac{\dot{\mu}x^2\mu+x^2\dot{z}z+3\,\dot{x}x^{5/3}-4\,\dot{y}y}{x^{2/3}}.
\end{aligned} \tag{18}$$

The Lagrangian

$$\mathcal{L}_D=\frac{1}{2}xu_x{}^2-\frac{1}{2}x^{5/3}u_y{}^2+\frac{1}{2}\frac{u_z{}^2}{\sqrt[3]{x}}+\frac{1}{2}\frac{u_\mu{}^2}{\sqrt[3]{x}}+\frac{1}{2}xV\left(x,y,\mu,z\right)u^2$$

generates the Klein-Gordon equation of the Petrov type D space,

$$-u_{xx}-x^{-1}u_x+x^{2/3}u_{yy}-u_{zz}x^{-4/3}-u_{\mu\mu}x^{-4/3}+V\left(x,y,\mu,z\right)u=0. \tag{19}$$

Finally, for this case, we have the potentials in Table 8.

Table 7.

Lie Brackets

$[Z^i, Z^j]$	Z^1	Z^2	Z^3	Z^4	Z^5
Z^1	0	0	0	0	$\frac{4Z^1}{3}$
Z^2	0	0	0	-Z^3	$\frac{Z^2}{3}$
Z^3	0	0	0	Z^2	$\frac{Z^3}{3}$
Z^4	0	Z^3	-Z^2	0	0
Z^5	$-\frac{4Z^1}{3}$	$-\frac{Z^2}{3}$	-$\frac{Z^3}{3}$	0	0

6. CONCLUSIONS

The main result of this paper is that we have derived the potentials of a Klein-Gordon equation admitted by some homothetic groups. We investigated Petrov spaces that are algebraically special vacuum spacetimes with homothetic vectors acting simply transitively. Namely, we have taken the homothety groups admitted by Petrov type II, III, N and D spacetimes.

It is well known that the Killing vector fields are a subset of the Noether point symmetries of a geodesic or classical Lagrangian. Under this consideration, note that we have deliberately ignored the Noether point symmetries that are dependent on the geodesic or arclength parameter s, as a symmetry of this type is irrelevant to our study. Note that the case of zero potential renders the Klein-Gordon equation to that of the wave equation and admits the vector fields $\mathcal{A}_i \cap P_{(u)} \cap P_{(\infty)}$. On the other hand, an arbitrary constant potential admits the same algebra as that of the wave equation, but excludes the homothetic vector in each case. An important counterpart of this analysis is that invariant solutions can be obtained corresponding to each potential found. This is an easy exercise and for a simple example: corresponding to the potential $V(x^2+y^2)$ in the Petrov type N case, the group invariant solution for Eq. (15) is $u(x,y,\mu,\nu) = \omega(\kappa)$ where $\kappa = x^2+y^2$ and ω satisfies the ordinary differential equation

$$4\omega_{\kappa,\kappa}\kappa + 4\omega_\kappa + V(\kappa)\omega = 0.$$

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7. APPENDIX

A. LIE AND NOETHER POINT SYMMETRIES

In this appendix, we briefly discuss the basic properties and definitions of the symmetries of differential equations. Consider a system with q unknown functions

Table 8.

The Homothetic Algebra $\mathcal{A}_4$ and Potential Functions

Potential Function $V(x,y,\mu,z)$	**Lie Symmetry or Algebra**	**Noether Symmetry or Algebra**
$V(x,y,\mu,z)$	$P_{(u)}$	$\times$
$V(x,\mu,z)$	Z^1	$\surd$
$V(x,y,z)$	Z^2	$\surd$
$V(x,y,\mu)$	Z^3	$\surd$
$V\left(x,y,\mu^2+z^2\right)$	Z^4	$\surd$
$\frac{1}{x^2}V\left(yx^{-4/3},\mu x^{-1/3},zx^{-1/3}\right)$	Z^5	$\surd$
$V(x)$	$\langle Z^1,Z^2,Z^3\rangle$	$\surd$
$x^{-2}V\left(zx^{-1/3}\right)$	$\langle Z^1,Z^2,Z^5\rangle$	$\surd$
$x^{-2}V\left(\mu x^{-1/3}\right)$	$\langle Z^1,Z^3,Z^5\rangle$	$\surd$
$x^{-2}V\left(\left(\mu^2+z^2\right)x^{-2/3}\right)$	$\langle Z^1,Z^4,Z^5\rangle$	$\surd$
$x^{-2}V\left(yx^{-4/3}\right)$	$\langle Z^2,Z^3,Z^5\rangle$	$\surd$
$V\left(x,\mu-\frac{by}{a},z\right)$	aZ^1+bZ^2	$\surd$
$V\left(x,\mu,z-\frac{by}{a}\right)$	aZ^1+bZ^3	$\surd$
$V\left(x,\mu^2+z^2,\frac{by-\arctan(\mu/z)a}{b}\right)$	aZ^1+bZ^4	$\surd$
$x^{-2}V\left(\frac{1}{4}\frac{4by+3a}{x^{4/3}b},\mu x^{-1/3},zx^{-1/3}\right)$	aZ^1+bZ^5	$\surd$
$V\left(x,y,z-\frac{b\mu}{a}\right)$	aZ^2+bZ^3	$\surd$
$V\left(x,y,-\frac{1}{2}\frac{b(\mu^2+z^2)+2az}{b}\right)$	aZ^2+bZ^4	$\surd$
$x^{-2}V\left(yx^{-4/3},\frac{b\mu+3a}{x^{1/3}b},zx^{-1/3}\right)$	aZ^2+bZ^5	$\surd$
$V\left(x,y,\frac{b(\mu^2+z^2)-2a\mu}{b}\right)$	aZ^3+bZ^4	$\surd$
$x^{-2}V\left(yx^{-4/3},\mu x^{-1/3},\frac{bz+3a}{x^{1/3}a}\right)$	aZ^3+bZ^5	$\surd$

u^a which depends on p independent variables x^i, i.e. we denote $u=(u^1,\dots,u^q)$ and $x=(x^1,\dots,x^p)$, respectively. Let

$$G_\alpha\left(x,u^{(k)}\right)=0,\quad \alpha=1,\dots,q, \tag{A.20}$$

be a system of m nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x. A one-parameter Lie group of transformations (ϵ is the group parameter) that is invariant under (A.20) is given by

$$\bar{x}=\Xi(x,u;\epsilon)\quad \bar{u}=\Phi(x,u;\epsilon). \tag{A.21}$$

Invariance of (A.20) under the transformation (A.21) implies that any solution $u = \Theta(x)$ of (A.20) maps into another solution $v = \Psi(x;\epsilon)$ of (A.20). Expanding (A.21) around the identity $\epsilon = 0$, we can generate the following infinitesimal transformations:

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon\xi^i(x,u) + \mathcal{O}(\epsilon^2), \quad i = 1,\dots,p, \\ \bar{u}^\alpha &= u^\alpha + \epsilon\eta^\alpha(x,u) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{A.22}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following infinitesimal vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{A.23}$$

The action of X is extended to all derivatives appearing in the equation in question through the appropriate prolongation. The infinitesimal criterion for invariance is given by

$$X\,[\text{LHS Eq.}(A.20)]\,|_{Eq.(A.20)} = 0. \tag{A.24}$$

The generalized total differentiation operator D_i with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots. \tag{A.25}$$

and W^α is the characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha\,. \tag{A.26}$$

The Euler-Lagrange equations, if they exist, is the system $\delta L/\delta u^\alpha = 0$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\,. \tag{A.27}$$

L is referred to as a Lagrangian. If we include point dependent gauge terms $f_1,\dots,f_n$, the Noether symmetries X are given by

$$X(L) + LD_i(\xi^i) = D_i(f_i). \tag{A.28}$$

Corresponding to each X, there exists a conserved vector $(T^1,\dots,T^n)$ that may then be determined by Noether's theorem

$$T^i = f^i - N^i(L). \tag{A.29}$$

where

$$D_i T^i = 0 \tag{A.30}$$

along the solutions of the differential equation. Here N^i is the Noether operator associated with the symmetry operator X given by

$$\mathbf{N^i} = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha} + \sum_{s\geq 1} D_{i_1} \dots D_{i_s} W^\alpha \frac{\delta}{\delta u^\alpha_{ii_1 \dots i_s}}, \tag{A.31}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by (A.27). The operator in Eq. (A.23) can be used to define the Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du}{\eta} = \dots$$

whose solution provides the invariant functions

$$W^{[r]}(x^i, u). \tag{A.32}$$

These invariants can be used in order to reduce the order of the PDE.

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