FAMILIES OF RATIONAL SOLITON SOLUTIONS OF THE KADOMTSEV–PETVIASHVILI I EQUATION

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Abstract. A family of exact explicit nonsingular rational soliton (lump) solutions of any order to the Kadomtsev–Petviashvili I equation are presented in a compact form. We show that the higher-order lump solutions may exhibit rich patterns on a finite background, but invariably evolve from a vertical distribution at large negative time into a horizontal distribution at large positive time, within an appropriate Galilean transformed frame. A universal polynomial equation is then put forward, whose real roots can accurately determine the lump positions in such a complex multi-lump distribution. We also unveil that there is an intimate relation between certain lump structures and the rogue-wave hierarchy. We expect that this finding may provide a new pathway for understanding the higher-dimensional rogue waves.

Key words: Rational soliton, rogue wave, Kadomtsev–Petviashvili equation.

1. INTRODUCTION

A resurgence of interest in classical integrable systems came with the discovery of solitons in the late 1960s, which were initially identified as the localized solutions of the Korteweg–de Vries (KdV) equation constructed via the method of inverse scattering transform (IST) [1, 2]. Now, integrable models significantly expand their territory and are equipped with a collection of well-established analytical approaches [3,4]. They have become the ideal platform for understanding those nonlinear entities ubiquitous in the physical world, such as solitons, breathers, and rogue waves [5–7]. The study of these solvable models is not only of fundamental interest but also has important practical implications in many areas of nonlinear physics.

As an integrable (2+1)-dimensional [(2+1)D] extension of the classic KdV equation, the Kadomtsev–Petviashvili (KP) equation [8] plays a fundamental role in nonlinear wave theory, allowing the formation of stable solitons or rational local-
ized solutions pertinent to systems involving quadratic nonlinearity, weak dispersion, and slow transverse variations [9]. This equation was originally proposed for modeling the ion-acoustic waves in plasmas [8] and the 2D shallow water waves [10]. Afterwards, it has been obtained as a reduced model in ferromagnetics, nonlinear optics, Bose-Einstein condensates, and string theory [11–15]. Particularly, in optical contexts, the KP equation and its generalizations have recently been employed in the study of collapse of ultrashort spatiotemporal optical pulses [16], and for the analysis of the dynamics of few-cycle optical solitons in diverse physical settings [17–20]. Most recently, numerical simulations showed that there is a close connection between the KP equation and the (2+1)D nonlinear Schrödinger (NLS) equation, thus further extending its application domains [21].

Besides the above physical origin, the KP equation has also attracted increasing mathematical interest since its inception [22–32]. It turns out that this equation has much richer solution structures than its 1D cousin—the KdV equation. For instance, depending on the transverse spatial characteristics, it involves the algebraically decaying lumps [22, 24–30] or the resonant line-solitons with complicated cluster algebras [31, 32]. Importantly, along with the study of the solution dynamics, an array of basic tools such as IST [22], Darboux transformation [23], Hirota bilinear method [24], Wronskian representation [25], and Painlevé analysis [26] were established. Fundamentally, many other known integrable systems can be obtained as reductions of the KP equation [9, 31].

Basically, the KP equation has two distinct versions, which bear a similarity in form but differ significantly in their underlying mathematical structure and the corresponding solution dynamics [9]. In this work, we are only concerned with the KP-I equation which can be written, in normalized form, as

\[ 3u_{yy} = (u_t + 6uu_x + u_{xxx})_x, \]  

where \( u(t, x, y) \) is a scalar function describing a nonlinear long wave of small amplitude propagating along the time \( t \), but with an asymmetric dependence on the spatial coordinates \( x \) and \( y \). Subscripts \( t, x, \) and \( y \) stand for the partial derivatives. It should be pointed out that the above interpretation of dependent variable \( u \) and of the coordinate variables can be adapted to the particular applicative contexts [11–13, 21, 33]. In addition, by rescaling variables, Eq. (1) can be transformed to other equivalent forms with different coefficients in front of each term; for instance, letting \( u \to -u \) and \( t \to -t \), Eq. (1) can be changed to the form shown in Ref. [21].

The search for the lump solutions to the KP-I equation is a long-standing issue and was extensively studied over the past decades. Up to now, only a limited number of original nonsingular lump solutions were reported [22, 27–30], mostly concentrating on the zero background scenarios and providing explicit solution forms up to the fourth order. In this paper, we present explicitly a family of nonsingular lump solu-
tions that propagate on a finite background, with the order extended to infinity. Such a treatment could be reminiscent of the rogue-wave solutions that are also required to localize on a nonzero background [6, 33]. We will show that these compact solutions enable one to gain an insight into the complicated multi-lump spatiotemporal dynamics, especially the unusual interaction dynamics. Furthermore, in virtue of these lump solutions, we find a universal polynomial equation for accurately determining the lump location in the multi-lump distribution. In addition, the connection to the rogue-wave hierarchy under a certain parameter condition is also discussed.

2. DARBOUX TRANSFORMATION AND EXACT RATIONAL SOLUTIONS

Before proceeding with the rational solutions, let us mention two basic properties related to the KP-I equation. First, we point out that Eq. (1) is invariant with respect to the Galilean transformation [26]

\[ x \rightarrow \chi \equiv x - 2ky + 12k^2t, \]
\[ y \rightarrow \rho \equiv y - 12kt, \]
\[ t \rightarrow t, \quad \text{and} \quad u(t, x, y) \rightarrow u(t, \chi, \rho), \]

where \( k \) is an arbitrary real parameter. In other words, if \( u(t, x, y) \) is a solution of Eq. (1), so is \( u(t, \chi, \rho) \), which can be obtained by simply replacing the original variables \( x \) and \( y \) with \( \chi \) and \( \rho \), respectively. As we will see, this property is very helpful for finding the general solutions.

Second, one can recall that the KP-I equation (1) can be cast into a linear eigenvalue problem, with the Lax pair defined through [23]

\[ i\psi_y + \psi_{xx} + u\psi = 0, \]
\[ \psi_t + 4\psi_{xxx} + 6u\psi_x + w\psi = 0, \]

where \( u \) is the scattering potential, \( \psi \) is the eigenfunction, and \( w \) is defined by \( w_x = 3uu_{xx} - 3iu_y \). Obviously, the compatibility of Eqs. (5) and (6) requires that the potential \( u \) must satisfy Eq. (1). This feature has implications in that by virtue of the Lax pair (5) and (6), the binary Darboux transformation [23, 29] can be conducted to seek for the new solution from the old one.

At variance with previous studies that used a zero seed [23, 29], we assume the initial solution to be \( u = a \) (\( a \neq 0 \)). Namely, we target to study the existence of lump solutions on a finite background. In this case, we let \( w = 0 \), and find that the corresponding eigenfunction \( \psi \) can take the form

\[ \psi = \exp[\Phi(t, x, y)], \]

with

\[ \Phi = \kappa x + i(\kappa^2 + a)y - 2(2\kappa^3 + 3a\kappa)t + \theta(\kappa), \]
where $\theta$ is a complex function of the spectral parameter $\kappa$. Here without loss of generality, we assume $\kappa$ to be real, as any imaginary part of $\kappa$ can be removed by the Galilean transformation defined by Eqs. (2)–(4).

As a consequence, using the repeated binary Darboux transformation associated with the Lax pair (5) and (6), one can obtain the $n$th-order solution $u^{(n)}$ of Eq. (1) and the corresponding eigenfunction $\psi^{(n)}$:

$$u^{(n)} = u + 2\ln(\det M)_{xx}, \quad \psi^{(n)} = \frac{\det M'}{\det M},$$

where $\det M (M')$ denotes the determinant of the matrix $M (M')$. Usually, there are two ways to define $M$, and hence $M'$. One convenient choice is to use the differential form called Wronskian representation [25]

$$\det M = W(\psi_1, \psi_2, \cdots \psi_n),$$

$$\det M' = W(\psi_1, \psi_2, \cdots \psi_n, \psi),$$

where $\psi_j$ ($j = 1, 2, \cdots, n$) are the linearly independent solutions of Eqs. (5) and (6) for given seed $u$ at specific $\kappa = \kappa_0$, and $W$ is the Wronskian defined by $W = \det A$, with the matrix element $A_{jm} = \frac{\partial m - 1 \psi_m}{\partial \kappa}$ [23]. However, this approach may result in complex and singular solutions that are physically irrelevant [34, 35], unless the functions $\psi_j$ are carefully constructed so that $M$ is positive definite, for example, separating each $\psi_j$ into two according to Eqs. (13) and (14) in Ref. [36].

The other efficient way is to use the integration form proposed in [29] for the matrix $M$ that will be intrinsically positive definite. Usually, $M$ has many possible constructions in terms of integrals [29]. For the purpose of unveiling the intriguing multi-lump dynamics, we will adopt the simplest integration form, viz.,

$$\det M = \int^x |\psi_n|^2 dx',$$

$$\det M' = \psi \int^x |\psi_n|^2 dx' - \psi_n \int^x \psi^\dagger \psi_n dx',$$

where

$$\psi_m = \frac{\partial^m \psi}{\partial \kappa^m} |_{\kappa=\kappa_0} = \ell_m \psi,$$

with $m = 1, 2, \cdots, n$ and $\psi$ given by Eq. (7). The dagger in Eq. (13) indicates the complex conjugate transpose and for the scalar function, $\psi^\dagger_m = \psi_m^*$. Obviously, $\ell_m$ will be a polynomial of degree $m$ in $t, x, y$ and satisfies the recursion relation

$$\ell_{m+1} = \frac{\partial \Phi}{\partial \kappa} \ell_m + \frac{\partial \ell_m}{\partial \kappa}, \quad \text{with} \quad \ell_0 = 1.$$
by substituting Eq. (12) into the first part of Eqs. (9), we obtain

\[ u^{(n)} = a + 2(\ln F_n)_{xx}, \quad (16) \]

with

\[ F_n = \sum_{j=0}^{2n} \frac{(-1)^j}{(2\kappa)^j} \frac{\partial^j}{\partial x^j} |\ell_n|^2. \quad (17) \]

We will show that the \( n \)th-order solution (16) with the high-order derivative polynomial (17) can be further simplified. For this end, we define

\[ \frac{\partial \Phi}{\partial \kappa} = x + 2i\kappa y - 6(2\kappa^2 + a)t + \gamma_1 \equiv \xi, \quad (18) \]
\[ \frac{\partial^2 \Phi}{\partial \kappa^2} = 2iy - 24\kappa t + \gamma_2 \equiv \vartheta, \quad (19) \]
\[ \frac{\partial^3 \Phi}{\partial \kappa^3} = -24t + \gamma_3 \equiv \eta, \quad (20) \]
\[ \frac{\partial^m \Phi}{\partial \kappa^m} = \gamma_m, \quad (m \geq 4). \quad (21) \]

Then, the polynomials \( \ell_j \) can be expressed in terms of the new variables \( \xi, \vartheta, \eta \), and the complex structural parameters \( \gamma_j \), with the help of the recursion relation (15). The first seven polynomials are presented as below:

\[ \ell_0 = 1, \quad \ell_1 = \xi, \quad \ell_2 = \xi^2 + \vartheta, \quad (22) \]
\[ \ell_3 = \xi^3 + 3\xi \vartheta + \eta, \quad (23) \]
\[ \ell_4 = \xi^4 + 6\xi^2 \vartheta + 4\xi \eta + 3\vartheta^2 + \gamma_4, \quad (24) \]
\[ \ell_5 = \xi^5 + 10\xi^3 \vartheta + 10\xi^2 \eta + 5\xi(3\vartheta^2 + \gamma_4) + 10\vartheta \eta + \gamma_5, \quad (25) \]
\[ \ell_6 = \xi^6 + 15\xi^4 \vartheta + 20\xi^3 \eta + 15\xi^2(3\vartheta^2 + \gamma_4) + 6\xi(10\vartheta \eta + \gamma_5) \]
\[ + 15\vartheta(\vartheta^2 + \gamma_4) + 10\eta^2 + \gamma_6. \quad (26) \]

As a result, after some algebraic manipulations, the formula \( F_n \) in Eq. (17) can be explicitly given by

\[ F_n = \sum_{k=0}^{n} \frac{[k!(\frac{\epsilon}{k})]^2}{\epsilon^{2k}} \left| \sum_{m=0}^{n} \frac{(-1)^m m! (\frac{\kappa + m}{\kappa}) \left( \frac{n-k}{m} \right) \ell_{n-k-m}}{\epsilon^m} \right|^2, \quad (27) \]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the binomial coefficient and \( \epsilon = 2\kappa \). It is now self-evident that \( F_n \) is always a positive polynomial of \( 2n \) degree, expressed by a sum of \( n + 1 \) absolute squares. Naturally, substitution of Eq. (27) into Eq. (16) yields the \( n \)th-order lump solution. The resultant general solution \( u^{(n)}(t, x, y) \) follows after trivial replacement operations \( x \rightarrow \chi \) and \( y \rightarrow \rho \), with \( \chi \) and \( \rho \) given by Eqs. (2) and (3).
We need to emphasize that Eq. (27) is the central result of this work, by which the lump solutions of arbitrary order can be obtained in a straightforward way. For later use, we expand the first six polynomials of $F_n$ as below:

$$F_1 = \left| \frac{\xi}{\epsilon} - \frac{1}{\epsilon^2} + \frac{1}{\epsilon^2} \right|^2,$$

$$F_2 = \left| \frac{\xi^2 + \vartheta - \frac{2}{\epsilon} \xi + \frac{2}{\epsilon^2}}{\epsilon^2} + \frac{4}{\epsilon^2} \left| \frac{\xi}{\epsilon} - \frac{2}{\epsilon^2} + \frac{4}{\epsilon^2} \right|^2,$$

$$F_3 = \left| \frac{\xi^3 + 3 \xi \vartheta + \eta - \frac{3}{\epsilon} (\xi^2 + \vartheta) + \frac{6}{\epsilon^2} \xi - \frac{6}{\epsilon^3} \right|^2,$$

$$+ \frac{9}{\epsilon^2} \left| \frac{\xi^2 + \vartheta - \frac{4}{\epsilon} \xi + \frac{6}{\epsilon^2}}{\epsilon^2} + \frac{36}{\epsilon^4} \left| \frac{\xi}{\epsilon} - \frac{3}{\epsilon^2} + \frac{36}{\epsilon^6} \right|^2,$$

$$F_4 = \left| \epsilon - \frac{4}{\epsilon} \ell_3 + \frac{12}{\epsilon^2} \ell_1 - \frac{24}{\epsilon^3} \ell_2 \right|^2 + \frac{4}{\epsilon^2} \left| \ell_1 - \frac{6}{\epsilon^2} \ell_2 + \frac{18}{\epsilon^3} \ell_1 - \frac{24}{\epsilon^4} \right|^2,$$

$$+ \frac{12^2}{\epsilon^4} \left| \ell_1 - \frac{6}{\epsilon} \ell_2 + \frac{12}{\epsilon^2} \right|^2 + \frac{24^2}{\epsilon^6} \left| \ell_1 - \frac{4}{\epsilon} \right|^2 + \frac{24^2}{\epsilon^8},$$

$$F_5 = \left| \epsilon - \frac{5}{\epsilon} \ell_4 + \frac{20}{\epsilon^2} \ell_3 - \frac{60}{\epsilon^3} \ell_2 + \frac{120}{\epsilon^4} \ell_1 - \frac{120}{\epsilon^5} \right|^2 + \frac{5}{\epsilon^2} \left| \ell_4 - \frac{8}{\epsilon} \ell_3 \right|^2,$$

$$+ \frac{36}{\epsilon^2} \ell_2 - \frac{96}{\epsilon^3} \ell_1 + \frac{120}{\epsilon^4} \ell_1 + \frac{20}{\epsilon^4} \ell_3 - \frac{9}{\epsilon} \ell_2 + \frac{36}{\epsilon^2} \ell_1 - \frac{60}{\epsilon^3} \right|^2,$$

$$+ \frac{60^2}{\epsilon^6} \left| \ell_2 - \frac{8}{\epsilon} \ell_1 + \frac{20}{\epsilon^2} \right|^2 + \frac{120^2}{\epsilon^8} \left| \ell_1 - \frac{5}{\epsilon} \right|^2 + \frac{120^2}{\epsilon^{10}},$$

$$F_6 = \left| \ell_6 - \frac{6}{\epsilon} \ell_5 + \frac{30}{\epsilon^2} \ell_4 - \frac{120}{\epsilon^3} \ell_3 + \frac{360}{\epsilon^4} \ell_2 - \frac{720}{\epsilon^5} \ell_1 + \frac{720}{\epsilon^6} \right|^2,$$

$$+ \frac{6^2}{\epsilon^2} \left| \ell_5 - \frac{10}{\epsilon} \ell_4 + \frac{60}{\epsilon^2} \ell_3 - \frac{240}{\epsilon^3} \ell_2 + \frac{600}{\epsilon^4} \ell_1 - \frac{720}{\epsilon^5} \ell_1 + \frac{720}{\epsilon^6} \right|^2,$$

$$+ \frac{7^2}{\epsilon^2} \ell_2 - \frac{240}{\epsilon^3} \ell_1 + \frac{360}{\epsilon^4} \ell_1 + \frac{360}{\epsilon^4} \ell_3 - \frac{120}{\epsilon^2} \ell_2 + \frac{60}{\epsilon^2} \ell_1 + \frac{120}{\epsilon^3} \right|^2,$$

$$+ \frac{360^2}{\epsilon^8} \left| \ell_2 - \frac{10}{\epsilon} \ell_1 + \frac{30}{\epsilon^2} \right|^2 + \frac{720^2}{\epsilon^{10}} \left| \ell_1 - \frac{6}{\epsilon} \right|^2 + \frac{720^2}{\epsilon^{12}}.$$

We should point out that the first four polynomials, Eqs. (28)-(31), have been reported in Refs. [27-29], with the assumption of $a = 0$. However, to the best of our knowledge, those given by Eqs. (32) and (33), and particularly the unified formula (27), which can extend the order to infinity, were not yet reported before. Of course, due to the specific form adopted for the matrix $M$ [see Eq. (12)], the obtained
polynomial (27) will not be the sole rational expression satisfying the KP-I equation (1) [29, 30].

Fig. 1 – Contour plots showing the spatial distribution at $t = 0$ for six types of lumps: (a) fundamental, (b) second-order, (c) third-order, (d) fourth-order, (e) fifth-order, and (f) sixth-order, obtained with $\alpha = 1$, $\epsilon = 1$, $k = 0$, $\gamma_1 = 1$, and $\gamma_2 = -10n$ (here $n$ is the lump order when applied, and other unshown $\gamma_j$ in each case are all set zero). (g) and (h) in the upper right corner show the 3D surface plots of (a) and (f), respectively.
3. MULTI-LUMP DYNAMICS AND DISCUSSION

A close inspection of the resultant \( n \)th-order lump solutions reveals that the lump structures are determined by three real parameters \( a, \epsilon, \) and \( k, \) and \( n \) complex structural parameters \( \gamma_j. \) Basically, while \( a \) determines the background height and can be normalized to unity, the free parameter \( \epsilon \) is closely related to the lump amplitude and peak position. As we will see, the free parameter \( k \) can be understood as the propagation constant along the \( x \) direction, which will lead the hump to tilt towards the \( y \) axis, while keeping the dips still along the \( x \) direction. Just as the name suggests, the structural parameters can substantially affect the spatiotemporal distribution of the lump structures.

Figure 1 shows the horizontally distributed lump solutions at \( t = 0 \), starting from the first order (fundamental) to the sixth order [see panels (a)–(f)], with \( a = 1, \) \( \epsilon = 1, \) and \( k = 0, \) but with a special choice of structural parameters, i.e., \( \gamma_1 = 1, \) and \( \gamma_2 = -10n \) (\( n \) is the lump order when applied). The other unshown structural parameters are all set to zero. With the above parameter settings, we see that there appear multiple lumps situated at \( y = 0, \) with almost constant intervals and the number of the humps equal to the lump order. We will show in the next Subsection that this interesting structural dynamics can be predicted in a deterministic way. Besides, we also see that the peak amplitude of the lumps increases as \( x \) runs from its negative side to its positive side [see, e.g., Figs. 1(f) and 1(h)].

In fact, the lumps can exhibit much more intriguing patterns than those shown in Fig. 1, which can be significantly affected by the structural parameters. Let us take for example the sixth-order lump solutions. Figure 2 shows the four types of sixth-order lump structures at \( t = 0, \) obtained with different sets of structural parameters, which have been specified in the caption, while keeping \( a, \epsilon, \) and \( k \) the same as in Fig. 1. When viewed along the \( y = 0 \) axis, one can clearly see that, besides the horizontal distribution in Fig. 1(f), these six lump components can exhibit an arc [see Fig. 2(a)], a vertical [see Fig. 2(b)], a Y-shaped [see Fig. 2(c)], or a square array [see Fig. 2(d)] distribution, depending on the specific choice of the structural parameters \( \gamma_j \) \((j = 1, 2, \ldots, 6). \) Basically, to obtain regular or symmetric contour distributions, the structural parameters should be taken real, as in Figs. 1 and 2.

3.1. A FIRST GLIMPSE INTO THE ABNORMAL EVOLUTION

Further inspection on the evolution of these lump structures with the time \( t \) reveals that even for a given set of structural parameters, the lump distribution in the plane \((x, y)\) may vary significantly with \( t. \) Figure 3(a) shows the isosurface plot of the sixth-order lump given in Fig. 1(f) with identical parameters and \( k = 0, \) clearly indicating that with the increase of time, its six components can change from a vertical distribution [see Fig. 3(c)] into a horizontal distribution [see Fig. 3(d)].
intriguing evolution can be seen by the $x$-$t$ view in Fig. 3(b), where at large negative time, the lump distribution tends to be perpendicular to the $(x, t)$ plane, while at large positive time, it will be parallel to the $(x, t)$ plane, as suggested by the green arrows in Fig. 3(a). We should point out that for a nonzero $k$ value, the lump distribution at $t = \pm \infty$ is still horizontally or vertically aligned, but in the $(\chi, \rho)$ plane. If converting back to the original $(x, y)$ plane, the lump distribution will be obliquely aligned parallel to the straight line $x = 2ky$ at $t = -\infty$, but still horizontally aligned at $t = +\infty$. This is a trivial result of the Galilean transformation invariance of Eq. (1) and we do not illustrate it here for brevity.

More interestingly, independently of the choice of structural parameters, the lump components always tend to locate parallel to the $\chi = 0$ direction at infinite negative time but along the $\rho = 0$ direction at infinite positive time. For illustration, we show in Fig. 4 the corresponding isosurface plots of the sixth-order lumps given in Fig. 2 within the time interval $[-20, 20]$, which are obtained with different sets of structural parameters. However, it is clearly seen that despite their diversity, all these lump structures have a similar propensity as shown in Fig. 3(a), namely, they tend to have a vertical distribution at $t = -\infty$ but a horizontal distribution at $t = +\infty$. 

![Image of plots showing lump distributions](http://www.infim.ro/rrp)
Fig. 3 – (a) Isosurface plot of the sixth-order lump given in Fig. 1(f) at $u = 1.2$, with its $x$-$t$ view being shown in (b). (c) and (d) demonstrate the lump structures at $t = -15$ and $15$, respectively, as compared to the lump structure shown in Fig. 1(f) with $t = 0$.

Fig. 4 – Isosurface plots (a)–(d) at $u = 1.2$ showing the evolutions of the sixth-order lumps with the time $t \in [-20, 20]$, corresponding to (a)–(d) in Fig. 2 under identical parameter conditions.
The case in Fig. 4(d) displays also such a propensity, although a little bit slower than the former three cases. This abnormal transition is contrary to what might be normally expected in soliton collisions [5] where the horizontally propagating solitons could still propagate horizontally after a collision. We note also that the two-lump interaction demonstrated in Ref. [30] is a normal process where an $x$ (or $y$)-aligned lump distribution is still $x$ (or $y$)-aligned after the collision (see Figs. 1 and 5 therein), distinctly different from the case under study.

The asymptotic behaviors of higher-order lumps at sufficiently large negative or positive $t$ can be understood from the original definitions (18)–(20) for the variables $\xi$, $\vartheta$, and $\eta$ that are solely involved in the polynomial (27). It is easy to verify that the lump distribution at $t = e$ (here $|e| \gg 1$) for a given set of structural parameters is indeed asymptotically equivalent to the one at $t = 0$ with the dominant parameters $\gamma_2 = -12e$ and $\gamma_3 = -24e$. As seen in Figs. 1(f) and 2(b), such kinds of structural parameters could produce a lump distribution parallel to the $\rho = 0$ or the $\chi = 0$ direction, of course excluding the fundamental lump situation in which only one lump is involved. Noting that the above conclusion is given in connection with the general $k$ value, a direct result of the aforementioned Galilean transformation property.

Several interesting conclusions can be drawn now. First, the $n$th-order lump solutions can be separated into at most $n$ lumps, whose peak amplitudes are higher in the positive $x$ regime than in the negative $x$ regime, see for instance Figs. 1(h), 2(c) and 2(d). Second, even for a given set of initial parameters, the lump structure, of course, excluding the fundamental one, may vary with the time and can evolve invariably from a vertical distribution at large negative $t$ into a horizontal distribution at large positive $t$, as seen in Figs. 3 and 4. This will also be true for a nonzero $k$ value, but in the coordinate frame $(t, \chi, \rho)$. Particularly, the energy exchange that takes place among lump components during their nonlinear interaction is also clearly apparent, when comparing Figs. 3(c) and 3(d). Last but not least, each lump component exhibits a Peregrine-soliton structure [37–40], namely, a hump flanked by two dips on a nonzero background, suggesting that it may be related to the rogue-wave prototype. However, contrarily to Peregrine solitons, the dips of each lump component do not necessarily go down to a zero amplitude, and the peak amplitude of the lump is not a fixed factor of the background height, both of which depend on the value of the free parameter $\epsilon$ [21]. The former two points follow from the multi-lump distribution dynamics and the last one is closely related to the rogue-wave phenomena. We will illuminate these interesting issues further in the next two subsections.

3.2. MULTI-LUMP DISTRIBUTION AND CENTRAL LOCATION

As a matter of fact, despite the complexity of the lump patterns, the central location of each lump component can be accurately given by the real roots of the
Fig. 5 – Second-order lumps consisting of two separated components given at $t = 0$, with peak locations compared to the analytical predictions based on Eq. (34), which are denoted by green plus signs in the contour plots (b), (d) and (f), for the same $\gamma_1 = 1 + 2i$ and different $\gamma_2$ values: (a, b) $\gamma_2 = -20 + 4i$, (c, d) $\gamma_2 = 20 + 4i$, and (e, f) $\gamma_2 = 20 + 40i$. The other parameters are the same as in Fig. 1(b), i.e., $\alpha = 1$, $\epsilon = 1$, and $k = 0$. 

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n-degree polynomial equation:
\[
\sum_{m=0}^{n} \frac{(-1)^m m! (n)}{e^m} \ell_{n-m} = 0.
\]
(34)

For simplicity, let us take as an example the second-order lump solution, which only
involves two complex structural parameters \(\gamma_1\) and \(\gamma_2\). For convenience, we separate
the structural parameters into the real and imaginary parts, i.e., letting \(\gamma_1 = \gamma_1^r + i\gamma_1^i\)
and \(\gamma_2 = \gamma_2^r + i\gamma_2^i\). Typically, we can let \(t = 0\) and \(k = 0\), as any nonzero values of \(t\)
(or \(k\)) can by removed by translation of the \(t\) axis (or by the Galilean transformation).
Then, the formula (34), by equating its real and imaginary parts to zero, respectively,
can be reduced to two coupled algebraic equations
\[
(x + \gamma_1^r - 1/\epsilon)^2 - \epsilon^2(y + \gamma_1^i/\epsilon)^2 + 1/\epsilon^2 + \gamma_2^r = 0,
\]
(35)
\[
2\epsilon(x + \gamma_1^r)(y + \gamma_1^i/\epsilon) + \delta = 0,
\]
(36)
where \(\delta = \gamma_2^r - 2\gamma_1^r/\epsilon\). Solving these two equations, one can obtain the central posi-
tions \((x_j, y_j)\) of two lump components. This can be done with two cases.

Case 1: As \(\delta = 0\), i.e., \(\gamma_2^r = 2\gamma_1^r/\epsilon\), one can solve readily Eqs. (35) and (36) for
the lump positions
\[
x_j = (-1)^j \sqrt{-1 - \gamma_2^r \epsilon^2/\epsilon + 1/\epsilon - \gamma_1^r}, \quad y_j = -\gamma_1^i/\epsilon,
\]
(37)
when \(\gamma_2^r \leq -1/\epsilon^2\), or
\[
x_j = -\gamma_1^r, \quad y_j = (-1)^j \sqrt{2 + \gamma_2^r \epsilon^2/\epsilon^2 - \gamma_1^i/\epsilon},
\]
(38)
as long as \(\gamma_2^r \geq -2/\epsilon^2\).

Case 2: Otherwise, if \(\gamma_2^r \neq 2\gamma_1^r/\epsilon\), we have
\[
x_j = -\frac{\delta}{2\epsilon \mu} - \gamma_1^r, \quad y_j = \mu - \gamma_1^i/\epsilon,
\]
(39)
where \(\mu\) is any of two real roots of the real-coefficient quartic equation
\[
\epsilon^2 \mu^4 - \left(\frac{2}{\epsilon^2} + \gamma_2^r\right) \mu^2 - \frac{\delta}{2\epsilon^2} \mu - \frac{\delta^2}{4\epsilon^2} = 0.
\]
(40)
According to the root criterion proposed in Ref. [41], the quartic equation (40) has
two distinct real roots only when its discriminant is smaller than zero. This condition
will be fully satisfied if the parameter relation \(\gamma_2^r > -7/(8\epsilon^2)\) holds true. In this case,
the two lump components can be well separated in the plane \((x, y)\).

For illustration, we show in Fig. 5 the second-order lump solutions using the
same \(\gamma_1 = 1 + 2i\) and different \(\gamma_2\) values, with the other initial parameters identical to Fig. 1(b). To be specific, we take \(\gamma_2 = -20 + 4i\) and \(\gamma_2 = 20 + 4i\) for the
top and middle plots in Fig. 5, respectively, both of which satisfy the condition
\[ \Im(\gamma_2 - 2\gamma_1/\epsilon) = 0, \]

while in the bottom plot we use the value \(\gamma_2 = 20 + 40i\). By inserting these values into Eqs. (37)–(39) with specific conditions being considered, one can readily obtain the peak positions of two lump components as \((\pm \sqrt{19}, -2)\), \((-1, \pm \sqrt{22} - 2)\), and \((-3.97, 4.06)\) and \((2.46, -7.20)\), which have been indicated by green plus signs in Figs. 5(b), 5(d), and 5(f), respectively. It is clear that our analytical predictions based on Eq. (34) about the central locations agree very well with the analytical solutions based on the polynomial (27). We can also see from Eqs. 5(e) and 5(f) that the lump distribution will no longer be symmetric if a set of complex structural parameters is adopted.

Calculations also show that the central positions shown in Figs. 1 and 2 almost coincide with those predicted by Eq. (34), under the corresponding parameter conditions. Usually, for the polynomial degree higher than three, there are no more explicit analytical formulas to describe these central positions, and one needs to resort to some routine root-finding schemes or commercial softwares. However, compared with the lengthy polynomial (27), Eq. (34) is much simpler and will be an efficient pathway to insight into the complicated multi-lump spatiotemporal dynamics.

### 3.3. GENUINE OR BOGUS ROGUE WAVES?

One may now wonder if these rational nonsingular solitons could be identified as rogue waves, as there is a striking similarity in their structures. As examples, the structure shown in Fig. 1(a) is reminiscent of the rogue wave prototype—Peregrine soliton [37–40], which is known to have a hump flanked by two holes. Likewise, the more complex structures shown in Fig. 5 are also found in coupled wave systems, where doublet rogue waves could take place [42–45]. Besides, the coexisting property of different lump structures on the same background, as shown in Fig. 2, can also be predicted to occur among rogue-wave structures [46]. To clarify this intriguing issue, let us take a closer look at the rational solutions defined by Eqs. (16) and (27). Here, for the sake of simplicity, only the fundamental lump structure is considered, and we examine further its profile as well as temporal evolution.

After insertion of Eq. (28) into Eq. (16) followed by the replacement \(x \rightarrow \chi\) and \(y \rightarrow \rho\), we obtain the general explicit fundamental lump solution:

\[
\begin{align*}
u = a + & \frac{4\{c^2 \rho^2 - [\chi - 3(c^2 + 2a)t]^2 + 1/c^2\}}{c^2 \rho^2 + [\chi - 3(c^2 + 2a)t]^2 + 1/c^2}, \\
\end{align*}
\]

(41)

where \(\chi\) and \(\rho\) are again defined by Eqs. (2) and (3). We note that in Eq. (41) the parameter \(\gamma_1\) has been removed by the coordinate translation so that the lump peak is located at the center. It is clear that this lump solution involves a free parameter \(c\) besides \(a\) and \(k\), and therefore involves a variable peak amplitude and two variable
minimums, which are given respectively by

\[ u_{\text{max}} = a + 4\epsilon^2, \quad u_{\text{min}} = a - \epsilon^2/2. \]  

(42)

Apparently, this feature does not match with the strict definition of the fundamental Peregrine soliton solution of the NLS equation, as the latter has a peak amplitude fixedly three times the background height and two side holes that can fall to zero in the dip center \([37, 40]\). In this regard, the solution (41) would resemble in dynamical structure a soliton more than a rogue wave, hence termed the lump in the early literatures \([24, 27, 28]\).

However, a recent work \([33]\) showed that the KP-I equation allows rogue-wave solutions as well. It suggests that Eq. (41) may be closely related to the Peregrine soliton profile. In fact, by letting \(\epsilon = \sqrt{2a}\), we find that the fundamental lump solution (41) can be expressed as the intensity of a typical Peregrine soliton, viz.,

\[ u = |p(t, x, y)|^2, \]  

(43)

where \(p(t, x, y)\) is the Peregrine soliton given by

\[ p = \sqrt{a} \left[ 1 - \frac{8ia\rho + 4}{2a(\chi - 12at)^2 + 4a^2\rho^2 + 1} \right] e^{i(kx + iy - 4\omega t)}, \]  

(44)

with the dispersion relations

\[ \iota = a - k^2, \quad \omega = 3ak - k^3. \]  

(45)

We further show that the latter can be the fundamental rogue-wave solution of the following combined complex modified KdV and NLS equations:

\[ p_t + 4p_{xxx} + 12|p|^2p_x = 0, \quad ip_y + p_{xx} + |p|^2p = 0, \]  

(46)

with \(k\) being the propagation constant along the \(x\) axis. Hence the relation (43) will be not surprising as the KP-I equation (1) can be derived from Eqs. (46) \([33]\).

Figure 6 illustrates this special fundamental lump structure \((\epsilon = \sqrt{2a})\) at \(t = 0\), with either \(k = 0\) or \(k = 1/2\). It is clear that the lump structure can take the shape of the typical Peregrine soliton intensity [see Figs. 6(a) and 6(b)], but exhibit an extra traveling wave behavior, as seen in Figs. 6(c) and 6(d). In Ref. \([33]\), we termed such a special structure a rogue-wave bullet in the context of nonlinear optics, by which we mean a nonlinear wave packet that has a characteristic rogue-wave profile in two dimensions, while it propagates without distortions in a third dimension, analogous to the X-shaped light bullet \([47–49]\) in a normally dispersive nonlinear medium \([50]\). Recent numerical simulations have confirmed that these rogue-wave bullets are stable to develop in spite of the onset of modulation instability \([33]\).
Fig. 6 – Fundamental lump solution (41) identified as the intensity of Peregrine solitons as \( \epsilon = \sqrt{2a} \) \((a = 1)\), for different propagation constants \((a, c) k = 0\) and \((b, d) k = 1/2\). The surface plots in \((a)\) and \((b)\) are given at \(t = 0\), while the isosurface plots shown in \((c)\) and \((d)\) are specified by \(u = 4\).

4. CONCLUSION

In conclusion, we presented in a compact form a family of exact explicit lump solutions of any order to the KP-I equation, which resemble rogue waves in characteristic profiles but with variable peak amplitude. Typically, we illustrated the unique spatiotemporal field distributions of sixth-order lump solutions under different parameter conditions. We found that the \(n\)th-order lump solution can be separated into at most \(n\) lumps, exhibiting rich patterns that depend on the choice of structural parameters. Nevertheless, even for a given set of initial parameters, the lump structure may vary with time and evolve invariably from a vertical distribution at large negative \(t\) into a horizontal distribution at large positive \(t\), within an appropriate Galilean transformed frame. We further showed that despite the complexity of the lump structures, each lump component in the spatiotemporal distribution can be located accurately by means of the polynomial equation (34). Additionally, in view of the great similarity between the lump structures and the rogue-wave entities, we studied the connection between these two kinds of rational solutions. It was revealed that under a certain parameter condition, the lump solutions can be identified as rogue waves, given that the KP-I equation can be derived from the combined \((1+1)D\) integrable equations (46), while the latter allow the rogue wave solutions [33]. Then, the overall dynam-
ics combining the prototypical two-dimensional rogue-wave profile with propagation in a third dimension supports the concept of rogue-wave bullets.

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