A PURELY QUARK LAGRANGIAN FROM QCD

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We present a method for determining a purely quark Lagrangian by mocking up the QCD partition function for large gauge couplings g. The resulting effective theory displays all the symmetries of low energy QCD and can be potentially used to explore hadron properties.

1. INTRODUCTION

Quantum chromodynamics is an $SU(3)$ gauge theory which contains quarks and gluons that interact with each other. Whereas at high energies the coupling constant is small and one can adequately describe the reality by expanding perturbatively in the coupling constant it is still not completely clear how one can depict the low-energy regime where the coupling constant is large and quarks form bound states of mesons and baryons. Various methods and Lagrangians have been proposed to describe the low energy dynamics: While some of them like the Nambu Jona Lasinio model [1], [2] are still based on fermions as fundamental degrees of freedom, most of them consider the mesons and baryons as the starting point. In the latter case one then constructs effective low-energy models which are widely based on the symmetries of the QCD Lagrangian like the $SU(3)_L \times SU(3)_R$ chiral symmetry, $U(1)_V$ or $U(1)_A$ (as well as assumptions about QCD vacuum and potential) embedded in the quark flavor sector of QCD. Models of low energy QCD with notable results regarding hadrons properties and interactions include chiral perturbation theory [3], [4], linear and nonlinear sigma models [5]- [22] or other symmetry induced effective Lagrangians [23]- [32]. Thus one can conclude that chiral symmetry whether spontaneously or explicitly broken is the major ingredient for building a low energy QCD effective theory.

In the present work however we shall consider a completely different point of view: we will not rely on symmetries but instead we shall build a purely dynamical Lagrangian and obtain the chiral symmetry as a derivative of the method. Our main tool is the QCD partition function from which one can derive all the properties of the particles and interactions. We start by considering an alternative description of the partition function after integrating out the quark degrees of freedom. As a consequence the original quarks will be replaced by copies (that describe the constituent quarks) with the same masses and quantum numbers but with different Lagrangian and thus interactions. In the end we shall obtain a Lagrangian where the gluon degrees of freedom have been eliminated and that contains only the quarks and their subsequent interactions. Note that our method implies only manipulation and calculations of the partition function and does not involve any loop computation.

We consider the effective fermion Lagrangian obtained in this paper as a purely theoretical one and as a first step towards a more comprehensive approach. The connection with the phenomenology of the bound states of mesons and baryons will be performed and finalized in a future work.

2. THE QCD LAGRANGIAN AND PARTITION FUNCTION

We start with the gauge fixed QCD Lagrangian with N colors and N_f flavors in the fundamental representation:

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{c}^a(-\partial^\mu\partial_\mu - g f^{abc}\partial^\mu A^b_\mu)c^c + \sum_f \bar{\Psi}_f(i\gamma^\mu D_\mu - m_f)\Psi_f, \tag{1}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{2}$$

and,

$$D_\mu = \partial_\mu - ig A^a_\mu t^a. \tag{3}$$

Here as usual t^a are the generators of the group $SU(N)$ in the fundamental representation.

We shall ignore the ghosts in what follows as they do not contribute essentially to our arguments. First we separate the Lagrangian in two pieces:

$$\begin{aligned}
\mathcal{L}_1 &= \sum_f \bar{\Psi}_f(i\gamma^\mu\partial_\mu - m_f + g\gamma^\mu t^a A^a_\mu)\Psi_f \\
\mathcal{L}_2 &= -\frac{1}{4}(F^a_{\mu\nu})^2.
\end{aligned} \tag{4}$$

The QCD partition function has the form:

$$Z_0 = \int dA^a_\mu(x) d\bar{c}^b(x) dc^d(x) d\bar{\Psi}_{fl} x d\Psi_{fl}(x) \exp[i\int d^4x \mathcal{L}], \tag{5}$$

where the usual rules of the path integration apply. Here f is the flavor index whereas l is the color one. Then one can integrate over the fermion variables to obtain:

$$Z_0 = \int dA^a_\mu(x) \prod_f \det\left[i\gamma^\mu \partial_\mu - m_f + g\gamma^\mu t^a A^a_\mu\right] \exp[\int d^4x \mathcal{L}_2]. \tag{6}$$

3. A SIMPLIFIED APPROACH

We shall discuss in particular the real life case of $N = 3$, $N_f = 3$ corresponding to QCD with three light flavors. We introduce the fermionic current that couples with the gluon field for each fermion species:

$$J^a_{\mu f} = \bar{\Psi}_f \gamma^\mu t^a \Psi_f. \tag{7}$$

For one generation of fermions there are 8×3 degrees of freedom for each fermion where 8 represents space time degrees of freedom of an off-shell fermion and 3 the number of colors. Overall we have $24N_f$ degrees of freedom for all fermions.

We consider a single flavor and next extend our arguments to N_f flavors. We start by making a change of variables from the elementary fermion degrees of freedom to the composite current J^a_μ. We first mention that in the case of anticommuting variables the Jacobian appear with an inverse power as compared to the case of commuting ones. Note that for this change of variables to make sense we need 24 degrees of freedom for J^a_μ instead of the 32 that one might obtain from a simple counting. This means that we need something similar to a gauge conditon which we choose to be $\partial^\mu J^a_\mu = \omega$ where ω is an arbitrary function as in the more standard case of a gauge field. Since there are exactly 8 constraints one obtains the desired matching of 24 degrees of freedom for both the fermion and vector boson variables. Then the change of variable is:

$$\int d\bar{\Psi} d\Psi \to \int dJ'^a_\mu \left| \frac{dJ^{a\prime}_\mu}{d\Psi_i} \right|, \tag{8}$$

where J'^a_μ is the subset of 24 components obtained from the variables J^a_μ after applying the constraints. The Jacobian in Eq. (8) is a determinant with dimension 24. Since any arbitrary derivative $\frac{dJ'^a_\mu}{d\Psi_{in}} = \bar{\Psi}_{jm}(\gamma_\mu)_{ji}(t^a)_{mn}$ (here the first indices are space time whereas the second ones are color) contains a fermion variable then the determinant will contain products of 24 fermion variables. Taking into account that there are exactly 24 distinct fermion variables and the anticommuting nature of these we conclude that the actual determinant will be given exactly by the product of the 24 distinct variables (since those term that contain a repeated variable will be zero) times an irrelevant constant factor. Thus:

$$\left| \frac{dJ^{a\prime}_\mu}{d\Psi_i} \right| = \text{const} \prod_{i,m} \Psi_{im}. \tag{9}$$

Then one can also write quite safely:

$$\left|\frac{dJ^{a\prime}_\mu}{d\Psi_i}\right| \rightarrow \delta(\partial^\mu J^a_\mu - \omega)\det[t^a\gamma^\mu J^a_\mu] \tag{10}$$

Here the determinant is taken in the space $\gamma^\mu t^a$ so it has the dimension 12 which leads exactly to a product of 24 distinct fermion variables. Note that at each point we take into account the composite nature of J^a_μ. Finally the complete change of variables takes the form:

$$\begin{aligned}&\int d\bar{\Psi}d\Psi \rightarrow dJ^a_\mu \delta(\partial_\mu J^a_\mu - \omega)\det[t^a\gamma^\mu J^a_\mu] \approx \\ &dJ^a_\mu \det[t^a\gamma^\mu J^a_\mu]\exp[-i\int d^4x \frac{(\partial^\mu J^a_\mu)^2}{2\xi}]\end{aligned} \tag{11}$$

The parameter ξ is arbitrary and we consider it very large so that the corresponding contribution in the exponential can be neglected. Note that the left hand side contains a product of measures over an even number of Grassmann variables which means that overall the left hand side is a real commuting quantity while the right hand side contains a product of measures over real variables since J^a_μ is real. Equation (11) is valid apart from a constant proportionality factor (which does not affect in any way the functional calculus since in computing correlators in QFT one always divides by the partition function).

We need to extend our arguments to $N_f = 3$ flavors. By applying the previous approach (or simply consider a product over the umber of flavors we obtain:

$$\int \prod_f d\bar{\Psi}_f d\Psi_f \rightarrow \int \prod_f dJ^a_{\mu f} \det[\gamma^\mu t^a J^a_{\mu f}]\delta\left(\partial^\mu J^a_{\mu f} - \omega\right). \tag{12}$$

It is more convenient however to write:

$$\prod_f \det[\gamma^\mu t^a J^a_{\mu f}] \rightarrow \text{const} \times \det[t^a\gamma^\mu \sum_f J^a_{\mu f}]^3 \tag{13}$$

The above relation is due to the fact that the above determinant contains a product of 72 fermion variables. But this is the total number of fermion variables so that each term that contains a variable twice is zero. Thus the determinant will be a constant times the product of 72 distinct fermion variables. But this is also what one would obtain from Eq. (12) so Eq. (13) is correct.

This being settled we need to write the full partition function in Eq. (6) in terms of the new variables $J^a_{\mu f}$. As we mentioned previously the change of variables makes sense only with the additional constraint on the field J^a_μ. We shall consider a particular case of it with $\omega(x) = 0$. On the other side this constraint is equivalent to constraining the free equation of motion of the fermion field. Thus it can be fulfilled only if,

$$i\gamma^\mu\partial_\mu\Psi - m\Psi = \frac{a(x)^2}{M}\Psi(x). \tag{14}$$

We use the function $\frac{a^2(x)}{M}$ (where M is an arbitrary scale) instead of simply $a(x)$ because we do not want to further constrain the fermion fields Ψ. Then the kinetic term for the fermion field will be replaced in the lagrangian by:

$$\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \frac{a^2(x)}{M}\bar{\Psi}\Psi. \tag{15}$$

Furthermore since the function $a(x)$ is arbitrary one can use the equation of motion to eliminate the kinetic term altogether from the Lagrangian.

The partition function in Eq. (6) will thus become:

$$Z_1 = \int dA^a_\mu \prod_f dJ^b_{\rho f} \det[t^a\gamma^\mu \sum_f J^a_{\mu f}]^3 \exp\left[i\int d^4x[g\sum_f J^a_{\mu f}A^{a\mu} + \mathcal{L}_2]\right] \tag{16}$$

The same partition function is obtained if one introduces three flavors of fermion copies χ_f, $\bar{\chi}_f$ of the original quarks such that:

$$Z_1 = \int dA^a_\mu \prod_{f'} dJ^b_{\rho f'} \prod_f d\bar{\chi}_f d\chi_f \exp\left[-i\int d^4x \frac{1}{M^2}\sum_f \bar{\chi}_f\gamma^\mu t^a[\sum_{f'} J^a_{\mu f'}]\chi_f\right] \times$$
$$\exp\left[i\int d^4x[\sum_{f'} J^a_{\mu f'} g A^{a\mu} + \mathcal{L}_2]\right]. \tag{17}$$

We make the change of variable $K^a_{\mu 1} = \sum_f J^a_{\mu f}$, $K^a_{\mu 2} = J^a_{\mu 2}$, $K^a_{\mu 3} = J^a_{\mu 3}$ to determine:

$$Z_1 = \int dA^d_\sigma dK^a_{\mu 1} K^b_{\rho 2} dK^c_{\nu 3} \prod_f d\bar{\chi}_f d\chi_f \exp\left[i\int d^4x \sum_f \frac{1}{M^2}\bar{\chi}_f\gamma^\mu t^a K^a_{\mu 1}\chi_f\right] \times$$
$$\exp\left[i\int d^4x[gK^a_{\mu 1}A^{a\mu} + \mathcal{L}_2]\right] =$$
$$\approx \int dA^a_\mu dK^b_{\rho 2} dK^c_{\nu 3} \delta\left(A^{a\mu} + \frac{1}{gM^2}\sum_f \bar{\chi}_f\gamma^\mu t^a\chi_f\right)\exp[i\int d^4x\mathcal{L}_2], \tag{18}$$

where the integral over $K^a_{\mu 1}$ is a delta function. Here one can drop the unwanted integrals over $K^b_{\rho 2}$ and $K^c_{\nu 3}$ since they do not contribute in any process and apply the delta function to obtain that the effective Lagrangian is just:

$$\mathcal{L}_{eff} = \mathcal{L}_2 \left(A^{a\mu} = -\frac{1}{gM^2} \sum_f \bar{\chi}_f \gamma^\mu t^a \chi_f \right). \tag{19}$$

Although this procedure seems oversimplifying it gives a correct glimpse of what kind of Lagrangian we should expect in terms of the current J^a_μ if the fermion kinetic term is neglected. The next section will contain a more general and comprehensive approach.

4. A COMPREHENSIVE APPROACH

We start with Eq. (6) which we rewrite here for completeness:

$$Z_0 = \int dA^a_\mu(x) \prod_f \det \left[i\gamma^\mu \partial_\mu - m_f + g\gamma^\mu t^a A^a_\mu \right] \exp \left[i \int d^4x \mathcal{L}_2 \right]. \tag{20}$$

We shall now try to reproduce the above partition function using a different set of variables.

We first need an identity that we shall prove in what follows. Consider the integral:

$$\begin{aligned} Z_x = &\int d\bar{w}_i dw_i d\bar{y}_j dy_j dJ_k dS_m \times \\ &\exp[i \int d^4x [\frac{1}{M} \bar{w} A^k w J^k + \bar{y} B^k y S^k + M^2 \sum_k S^k J^k]]. \end{aligned} \tag{21}$$

Here w_i, y_i and their conjugates are each a set of n fermion variables (we include in n the space time components) and J_k, S_k bare a set of regular scalars with mass dimension 1. The index k goes also from 1 to m and the matrices A^k, B^k are each a set of m $(n \times n)$ matrices where A^k have mass dimension 1 and B^k have mass dimension 0. We will solve first the integral over the fermion fields to get:

$$
\begin{aligned}
&Z_x = \int \prod_n dJ^n dS^n \det[i\frac{1}{M}\sum_k A^k J^k]\det[i\sum_m B^m S^m]\times \\
&\exp[i\int d^4x M^2 \sum_r J^r S^r] \approx \\
&\int \prod_n dJ^n dS^n \det[\frac{1}{M}\sum_{k,m} A^k B^m J^k S^m]\exp[i\int d^4x \sum_r M^2 J^r S^r] \approx \\
&\int \prod_n dJ^n dS^n d\bar{z}_i dz_i \exp[i\int d^4x[\bar{z}[\frac{1}{M^2}\sum_{k,m} A^k B^m J^k S^m]z + \sum_r M^2 J^r S^r]] = \\
&\int \prod_{k,i} dS^k d\bar{z}_i dz_i \delta(M^2 S^k + \bar{z}\frac{1}{M^2}\sum_m A^k B^m S^m z) \approx \\
&\int \prod_{k,i} dS^k d\bar{z}_i dz_i \delta(S^k(M^2 + \bar{z}\frac{1}{M^2}A^k B^k z) + \bar{z}\frac{1}{M^2}\sum_{m\neq k} A^k B^m S^m z) \\
&\int \prod_i d\bar{z}_i dz_i \prod_k \frac{1}{M^2 + \frac{1}{M^2}\bar{z}A^k B^k z}
\end{aligned}
\tag{22}
$$

Here in the third line we expressed the determinant as a path integral over an extra pair of n fermion variables z_i and $\bar{z}_i$ and we dropped all unimportant constant factors. In order to further process the result in Eq. (22) we consider a lattice in the coordinate euclidean space such that:

$$
\int d^4x \to = \frac{1}{\Lambda^4}\sum_{x_{En}}, \tag{23}
$$

where Λ is the cut-off corresponding to the lattice and $\sum_{x_{En}}$ goes over all coordinates and lattice points. Then:

$$
\begin{aligned}
&\prod_{k,n}\frac{1}{M^2+\frac{1}{M^2}\bar{z}A^kB^kz}=\\
&\exp[-\sum_{k,n}\ln[M^2+\frac{1}{M^2}\bar{z}A^kB^kz]]=\\
&\exp[-\sum_{k,n}\ln[M^2]-\sum_{k,n}\ln[1+\frac{1}{M^4}\bar{z}A^kB^kz]]=\\
&\exp[-\sum_{k,n}\ln[M^2]-\sum_{k,n}\frac{1}{M^4}[\bar{z}A^kB^kz]+\frac{1}{2}\sum_{k,n}[\frac{1}{M^4}\bar{z}A^kB^kz]^2+...]\approx\\
&\text{const}\times\exp[-\frac{\Lambda^4}{M^4}\int d^4x_E\sum_k\bar{z}A^kB^kz]=\\
&\text{const}\times\exp[i\int d^4x\sum_k\bar{z}A^kB^kz].
\end{aligned}
\tag{24}
$$

In the last line of Eq. (24) we considered $\Lambda=M$ and we neglected terms of order $\frac{1}{M^4}$ and higher. We also expressed everything in the Minkowski space through the substitution $x_0=ix_{E0}$. Then from Eqs. (22) and (24) we obtain:

$$
Z_x=\text{const}\int d\bar{z}_idz_i\exp[i\int d^4x\bar{z}A^kB^kz]=\text{const}\times\det[iA^kB^k].
\tag{25}
$$

Here we implicitly assume that repeating indices are summed over, convention which will be respected also in what follows.

Now consider again Z_x and this time we integrate first over J_k and S_k:

$$
\begin{aligned}
&Z_x=\int d\bar{w}_idw_id\bar{y}_jdy_j\prod_n dJ^ndS^n\times\\
&\exp[i\int d^4x[\frac{1}{M}\bar{w}A^kwJ^k+\bar{y}B^kyS^k+\sum_k M^2S^kJ^k]]\approx\\
&\int d\bar{w}_idw_id\bar{y}_jdy_j\prod_p dJ^p\frac{1}{M^2}\delta(J^p+\frac{1}{M^2}\bar{y}B^py)\exp[i\frac{1}{M}\bar{w}A^kwJ^k]\approx\\
&\text{const}\times\int d\bar{w}_idw_id\bar{y}_jdy_j\exp[-i\int d^4x\frac{1}{M^3}\bar{w}A^kw\bar{y}B^ky]\approx\\
&\text{const}\times\det[iA^mB^m].
\end{aligned}
\tag{26}
$$

The result in the last line of Eq. (26) will be of most importance in what follows.

There is an important extension to the results in Eqs. (22) and (26) which we will use here but state without proof because the proof is just a simple generalization of the arguments above:

$$
\begin{aligned}
&Z_z=\int d\bar{w}_i dw_i d\bar{y}_j dy_j \prod_n dJ^n dS^n \times \\
&\exp[i\int d^4x[\frac{1}{M}\bar{w}A^k wJ^k+\bar{y}B^k yS^k+M^2\sum_k S^k J^k+T(\bar{w},w)]]= \\
&=\int d\bar{w}_i dw_i d\bar{y}_j dy_j \exp[-i\int d^4x\frac{1}{M^3}[\bar{w}A^k w\bar{y}B^k y]]\approx \\
&\mathrm{const}\times\det[iA^m B^m].
\end{aligned}
\tag{27}
$$

Here T is any polynomial of the type $\sum_k(\bar{w}C_k w)^k$. This is true because both integrals over w and y must be satisfied simultaneously and the number of variables w that are integrated in the fermion path integral must match the number of variables y.

We thus start from:

$$
\begin{aligned}
&Z_x=\int d\bar{w}_i dw_i d\bar{y}_j dy_j \prod_n dJ^n dS^n \times \\
&\exp[i\int d^4x[\frac{1}{M}\bar{w}A^k wJ^k-\bar{y}B^k yS^k+\sum_k M^2 S^k J^k+T(w,\bar{w})]]= \\
&=\int d\bar{w}_i dw_i d\bar{y}_j dy_j \exp[i\int d^4x[\frac{1}{M^3}\bar{w}A^k w\bar{y}B^k y+T(w,\bar{w})]]\approx \\
&\mathrm{const}\times\det[A^k B^k],
\end{aligned}
\tag{28}
$$

and consider the set w_i as being $N\frac{N_f}{2}$ fermions Ψ_{if} , where N is the number of colors and $\frac{N_f}{2}$ is the number of flavors. Similarly the set y_i is a similar set of $N\frac{N_f}{2}$ fermions χ_{if}. In total we consider to have $N_f=6$ flavors of fermions corresponding to the 6 flavors of quarks of the standard. We further define the matrices A^k, B^k as:

$$
\begin{aligned}
&A^k=[i\partial_\mu t^a-mt^a\gamma_\mu+g' A^a_\mu] \\
&B^k=\gamma^\mu t^a
\end{aligned}
\tag{29}
$$

$$
T=\frac{1}{m_0^2}(\sum_f\bar{\Psi}_f\Psi_f-v^3)^2,
\tag{30}
$$

where v is a constant with mass dimension 1 and m_0 is an arbitrary scale. Then:

$$
\begin{aligned}
&A^k B^k==[i\gamma^\mu\partial_\mu t^a t^a-4mt^a t^a+g'\gamma^\mu t^a A^a_\mu]= \\
&[i\gamma^\mu\partial_\mu\frac{N^2-1}{2N}-4m\frac{N^2-1}{2N}+g'\gamma^\mu t^a A^a_\mu]= \\
&\frac{N^2-1}{2N}[i\gamma^\mu\partial_\mu-4m+g\gamma^\mu t^a A^a_\mu]
\end{aligned}
\tag{31}
$$

where we redefined $g' = \frac{N^2-1}{2N} g$ and $4m = m_f$. We observe that the operator $A^k B^k$ is the operator that appears in the standard model between two fermion states. Note that we can include in each A^k and B^k a diagonal flavor matrix with the same final result.

Then the counterpart of the Eq. (28) in terms of the above definition will be:

$$\begin{aligned}
&Z_0 = \int dA^a_\mu(x) \prod_f \det\left[i\gamma^\mu\partial_\mu - m_f + g\gamma^\mu A^a_\mu\right] \exp\left[i[\int d^4x\mathcal{L}_2 + \int d^4xT]\right] = \\
&= \int dA^a_\mu \prod_{f=1}^{3} d\bar\Psi_f d\Psi d\bar\chi_f d\chi_f \exp\left[i\int d^4x\mathcal{L}_2\right] \times \\
&\exp\left[i[\frac{1}{M^3}\int d^4x \sum_{f=1}^{3}[\bar\Psi_f(it^a\partial_\mu - mt^a\gamma_\mu + g'A^a_\mu)\Psi_f] \sum_{f'=1}^{3}[\bar\chi_{f'}\gamma^\mu t^a \chi_{f'}] + \right. \\
&\left.\int d^4xT]\right].
\end{aligned} \tag{32}$$

Here M is an arbitrary scale that reestablishes the correct dimensionality. All constants m_0, v and M with mass dimension 1 are arbitrary and should be determined subsequently from phenomenological arguments.

We denote:

$$\begin{aligned}
&Z_1 = \int \prod_{f=1}^{3} d\bar\Psi_f d\Psi_f d\bar\chi_f d\chi_f \times \\
&\exp\left[i[\frac{1}{M^3}\int d^4x \sum_{f=1}^{3}[\bar\Psi_f(it^a\partial_\mu - mt^a\gamma_\mu + g'A^a_\mu)\Psi_f] \sum_{f'=1}^{3}[\bar\chi_{f'}\gamma^\mu t^a \chi_{f'}] + \right. \\
&\left.\int d^4xT]\right],
\end{aligned} \tag{33}$$

and work only with it.

First we will make the change of variable $J^a_{\mu f} = \frac{1}{M^2}\bar\chi_f\gamma^\mu t^a\chi_f$ (see section II for details). Note that this implies a subsequent gauge condition for $J^a_{\mu f}$ which we shall discuss later. The partition function Z_1 will become:

$$Z_1 = \int \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f dJ^a_{\mu f} \det[J^a_{\mu f}\gamma^\mu t^a] \times$$
$$\exp\left[i[\frac{1}{M}\int d^4x \sum_{f=1}^{3} \bar{\Psi}_f (it^a\partial_\mu - mt^a\gamma_\mu + g' A^a_\mu)\Psi_f \sum_{f=1}^{3} J^{a\mu f} + \right.$$
$$\left. \int d^4xT] \right]. \tag{34}$$

Noting that in the change of variable we can use $\det[\sum_f J^a_{\mu f}\gamma^\mu t^a]^3$ instead of $\prod_f \det[J^a_{\mu f}\gamma^\mu t^a]^3$ (see Eq. (13) we can further write:

$$Z_1 = \int \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f d\Psi_f dJ^a_{\mu f} \det[\sum_f J^a_{\mu f}\gamma^\mu t^a]^3 \times$$
$$\exp\left[i[\frac{1}{M}\int d^4x \sum_{f=1}^{3} [\bar{\Psi}_f (it^a\partial_\mu - mt^a\gamma_\mu + g' A^a_\mu)\Psi_f] \sum_{f=1}^{3} J^{a\mu f} + \right.$$
$$\left. \int d^4xT] \right]. \tag{35}$$

We can further make a change of variables $Y^a_\mu = \sum_{f=1}^{3} J^a_{\mu f}$, $Z^a_\mu = J^a_{\mu 2}$, $U^a_{\mu 3} = J^a_{\mu 3}$ and drop the unwanted integrals over Z^a_μ and U^a_μ as they would not contribute to any process. This yields:

$$Z_1 \approx \int dY^a_\mu \int \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f \det[Y^a_\mu \gamma^\mu t^a]^3 \times$$
$$\exp\left[i[\frac{1}{M}\int d^4x \sum_{f=1}^{3} [\bar{\Psi}_f (it^a\partial_\mu - mt^a\gamma_\mu + g' A^a_\mu)\Psi_f] Y^{a\mu} + \int d^4xT] \right] =$$
$$\int dY^a_\mu \int \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f d\bar{\xi}_f d\xi_f \times$$
$$\exp\left[i[\frac{1}{M}\int d^4x \sum_{f=1}^{3} [\bar{\Psi}_f (it^a\partial_\mu - mt^a\gamma_\mu + g' A^a_\mu)\Psi_f] Y^{a\mu} + \right.$$
$$\left. a\sum_{f=1}^{3} \bar{\xi}_f \gamma_\mu t^a \xi_f Y^{a\mu} + \int d^4xT] \right]. \tag{36}$$

Here we introduced another set $\bar{\xi}_f$ and ξ_f of 3 fermions to account for the determinant in the first line of Eq. (36). Here a is an arbitrary dimensionless coupling constant

As we mentioned previously in order to be able to make a change of variable from the fermions χ_f to the currents $J^a_{\mu f}$ one would need something similar a to gauge condition to cut off the number of degrees of freedom from 32 to 24. We shall consider this constraint as $\partial^\mu J^a_\mu = 0$. Upon the change of variables to Y^a_μ, Z^a_μ and U^a_μ this constraint will become $\partial^\mu Y^a_\mu = 0$ for the only variable of interest. We shall introduce this in the partition function Z_1 as:

$$\delta(\partial^\mu Y^a_\mu) = \int dS^a \exp[i\int d^4x M_1 S^a \partial^\mu Y^a_\mu] = \int dS^a \exp[-i\int d^4x M_1 \partial^\mu S^a J^a_\mu], \quad (37)$$

where M_1 is an arbitrary constant with mass dimension 1. With the addition of the gauge condition the partition function in Eq. (36) will become:

$$\begin{aligned}
&Z_1 \approx \int dY^a_\mu dS^a \int \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f \det[Y^a_\mu \gamma^\mu t^a]^3 \times \\
&\exp\left[i[\frac{1}{M}\int d^4x \sum_{f=1}^{3}[\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu + g' A^a_\mu)\Psi_f]Y^{a\mu} + \int d^4x T]\right] = \\
&\int dY^a_\mu \int \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f d\bar{\xi}_f d\xi_f \times \\
&\exp\left[i\int d^4x[\frac{1}{M}\sum_{f=1}^{3}[\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu + g' A^a_\mu)\Psi_f]Y^{a\mu} + \right. \\
&\left. a\sum_{f=1}^{3}\bar{\xi}_f\gamma_\mu t^a \xi_f Y^{a\mu} - M_1\partial_\mu S^a Y^{a\mu} + T]\right]. \qquad (38)
\end{aligned}$$

We shall integrate $Y^{a\mu}$ by observing that it couples only linearly that the Lagrangian is hermitian and thus it leads to a product of delta functions. This yields:

$$\begin{aligned}
&Z_1 = \int dS^a \prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f d\bar{\xi}_f d\xi_f \times \\
&\delta(g' A^a_\mu \sum_f \bar{\Psi}_f\Psi_f + \sum_{f=1}^{3}[\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu)\Psi_f] + Ma\sum_{f=1}^{3}\bar{\xi}_f\gamma_\mu t^a\xi_f - M_1 M\partial_\mu S^a) \times \\
&\exp[i\int d^4x T] \qquad (39)
\end{aligned}$$

We shall regard the delta function in the space of variables A^a_μ.

Before going further we need to make an important ammendment. When we introduce the change of variable from χ_f, $\bar{\chi}_f$ to $J^a_{\mu f}$ we replaced (we consider here for simplicity only one fermion species):

$$\int d\bar{\chi}_f d\chi_f \approx \int dJ^a_{\mu f}\det[\gamma^\mu t^a J^a_{\mu f}] \tag{40}$$

The reason stems form the fact that when we transform from one set of variables one gets always products of 24 different fermion components. However one can extend the above transformation to include some function in the determinant. Consider that we have:

$$\int dJ^a_\mu \det[i\gamma^\mu\partial_\mu - m + kt^a\gamma^\mu J^a_\mu] \tag{41}$$

and we apply the inverse transformation to fermion variables χ_f:

$$\begin{aligned}&\int dJ^a_\mu \det[i\gamma^\mu\partial_\mu - m + kt^a\gamma^\mu J^a_{\mu f}] = \\ &\int d\bar{\chi}_f d\chi_f |\frac{dJ^a_\mu}{d\chi_f}|^{-1}\det[i\gamma^\mu\partial_\mu - m + kt^a\gamma^\mu\bar{\chi}_f\gamma_\mu t^a\chi_f]\end{aligned} \tag{42}$$

Moreover we also need to consider a function (for one flavor) to be integrated which is of the type $\det[X^a_\mu J^a_\mu]$. Assume we express this in terms of fermion components. Those variables that repeat themselves will get canceled. In the end $\det[X^a_\mu J^a_\mu] = f\bar{\chi}_1\bar{\chi}_2..\chi_1\chi_2$ where f is an arbitrary function and the product is over all 24 fermion components for one flavor (note that the product might contain derivative which we omit for simplicity). Then Eq. (42) will become:

$$\begin{aligned}&\int dJ^a_\mu \det[i\gamma^\mu\partial_\mu - m + kt^a\gamma^\mu J^a_{\mu f}]\det[X^a_\mu J^a_\mu] = \\ &\int d\bar{\chi}d\chi\frac{1}{\bar{\chi}_1\bar{\chi}_2...\chi_1\chi_2..}[a_0 + a_1\bar{\chi}_i\chi_j + ...a_{12}\bar{\chi}_1\bar{\chi}_2...\chi_1\chi_2..]\times[\bar{\chi}_1\bar{\chi}_2...\chi_1\chi_2..] = \\ &\int d\bar{\chi}d\chi[a_0 + a_1\bar{\chi}_i\chi_j + ...a_{12}[\bar{\chi}_1\bar{\chi}_2...\chi_1\chi_2..]] = \int d\bar{\chi}d\chi[a_{12}\bar{\chi}_1\bar{\chi}_2...\chi_1\chi_2..]\end{aligned} \tag{43}$$

But the end result corresponds to that part of $\det[i\gamma^\mu\partial_\mu - m + kt^a\gamma^\mu J^a_{\mu f}]$ that contains only J^a_μ so the addition we make to $\det[t^a\gamma^\mu J^a_\mu]$ is irrelevant. The main point of the above discussion is that even when we integrate over J^a_μ the intrinsic fermion nature of the variables should be considered. For most of the purposes we shall consider J^a_μ however as a regular commuting variable. However our discussion has a counterpart by discussing purely the J^a_μ's. This being settled one can add in Eq. (39)

a kinetic term for the fermions $\bar{\xi}_f$, ξ_f as in:

$$
\begin{aligned}
&Z_1 = \int\int\prod_{f=1}^{3} d\bar{\Psi}_f d\Psi_f d\bar{\xi}_f d\xi_f dS^a \times \\
&\prod \delta(g' A_\mu^a \sum_f \bar{\Psi}_f\Psi_f + \sum_{f=1}^{3}[\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu)\Psi_f] + Ma\sum_{f=1}^{3}\bar{\xi}_f\gamma_\mu t^a\xi_f - M_1\partial_\mu S^a)) \times \\
&\exp[i[\int d^4x[\sum_f[\bar{\xi}_f(i\gamma^\mu\partial_\mu - M_f)\xi_f] + \int d^4xT]]. \qquad (44)
\end{aligned}
$$

The expression in Eq. (44) is the final partition function we will work with.

It appears that the delta function in Eq. (44) is badly defined. In order to fix this we go back to the full Lagrangian of interest (including the pure gluon one) which is:

$$
\begin{aligned}
&\mathcal{L} = \sum_f[\bar{\xi}_f(i\gamma^\mu\partial_\mu - M_f)\xi_f] + T + \\
&J_\mu^a[g' A_\mu^a \sum_f \bar{\Psi}_f\Psi_f + \sum_{f=1}^{3}[\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu)\Psi_f] + Ma\sum_{f=1}^{3}\bar{\xi}_f\gamma_\mu t^a\xi_f - M_1M\partial_\mu S^a)] + \\
&\mathcal{L}_2(A_\mu^a, g). \qquad (45)
\end{aligned}
$$

We first recall that $g' = \frac{N^2-1}{2N}g$ and make the change of variables $gA_\mu^a \Rightarrow A_\mu^a$. Note that we can do this at any time without affecting in any way the dynamics. This yields (we use for simplicity the same notation):

$$
\begin{aligned}
&\mathcal{L} = \sum_f[\bar{\xi}_f(i\gamma^\mu\partial_\mu - M_f)\xi_f + T + \\
&J_\mu^a[\frac{N^2-1}{2N}A_\mu^a \sum_f \bar{\Psi}_f\Psi_f + \sum_{f=1}^{3}[\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu)\Psi_f] + Ma\sum_{f=1}^{3}\bar{\xi}_f\gamma_\mu t^a\xi_f - M_1M\partial_\mu S^a)] + \\
&\frac{1}{g^2}\mathcal{L}'_2(A_\mu^a), \qquad (46)
\end{aligned}
$$

where $\mathcal{L}'_2$ does not contain any more any trace of g. We are interested in the regime where g is large so we can consider $\frac{1}{g^2}$ a small parameter. Next we take into account that one can solve from the delta function A_μ^a as function of the other variables. With this substitution the Lagrangian becomes:

$$
\begin{aligned}
\mathcal{L} &= \sum_f[\bar{\xi}_f(i\gamma^\mu\partial_\mu - M_f)\xi_f] + T + \frac{1}{g^2}\mathcal{L}'_2(\bar{\xi}_f, \xi_f, \bar{\Psi}_f, \Psi_f, S^a) \\
\mathcal{L}_0 &= \sum_f[\bar{\xi}_f(i\gamma^\mu\partial_\mu - M_f)\xi_f] + T, \qquad (47)
\end{aligned}
$$

Assume we consider separately shifts in the variables ξ_f, Ψ_f and their subsequent conjugates:

$$\begin{aligned}\xi_f &= \xi_f + \frac{1}{g^2}\xi'_f \\ \bar{\xi}_f &= \bar{\xi}_f + \frac{1}{g^2}\bar{\xi}'_f \\ \Psi_f &= \Psi_f + \frac{1}{g^2}\Psi'_f \\ \bar{\Psi}_f &= \bar{\Psi}_f + \frac{1}{g^2}\bar{\Psi}'_f.\end{aligned} \tag{48}$$

Then one can still obtain the interaction Lagrangian of order $\frac{1}{g^2}$ provided that the fields Ψ_f and ξ_f satisfy the equation of motion for the free Lagrangian $\mathcal{L}_0$. Recalling the definition of T from Eq. (30) we get:

$$\frac{\partial T}{\partial \Psi_g} = \frac{2}{m_0^3}\Psi_g(\sum_f \bar{\Psi}_f\Psi_f - v^3) = 0 \tag{49}$$

whose only non trivial solution is $\sum_f \bar{\Psi}_f\Psi_f = v^3$. Thus the unwanted denominator in the δ function in Eq. (44) is conveniently fixed.

We redefine $\frac{1}{v^3}M = y\frac{1}{v^2}$ and $\frac{1}{v^3}MM_1 = z\frac{1}{v}$ where y and z are two adimensional coefficients.

From Eqs. (44) and (47) we obtain the effective fermion Lagrangian in terms of the two sets of fermions Ψ_f (the light quarks) and ξ_f (the heavy quarks):

$$\mathcal{L} = \sum_f[\bar{\xi}_f(i\gamma^\mu\partial_\mu - M_f)\xi_f] + \frac{1}{m_0^2}(\sum_f \bar{\Psi}_f\Psi_f - v^3)^2 + \frac{1}{g^2}\mathcal{L}'_2(A^a_\mu(\Psi_f,\xi_f,S^b)), \tag{50}$$

where,

$$\begin{aligned}&A^a_\mu(\Psi_f,\xi_f,S^b) = \\ &-\frac{2N}{N^2-1}[\frac{1}{v^3}\sum_{f=1}^3\bar{\Psi}_f(it^a\partial_\mu - mt^a\gamma_\mu)\Psi_f + y\frac{1}{v^2}\sum_{f=1}^3\bar{\xi}_f\gamma_\mu t^a\xi_f - z\frac{1}{v}\partial_\mu S^a]\end{aligned} \tag{51}$$

We can further refine Eq. (51) by taking A^a_μ equal to the real part of the right hand side which leads to the additional constraint on the field Ψ_f:

$$\partial_\mu(\sum_{f=1}^3\bar{\Psi}_f t^a\Psi_f) = 0. \tag{52}$$

In Eq. (50) the term $\mathcal{L}'_2$ is the pure gluon Lagrangian independent of the gauge coupling constant:

$$\mathcal{L}_2' = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}. \tag{53}$$

Note that in Eq. (51) the presence of the scalars S^a ($a = 1...8$) is purely optional as there are many ways in which one can implement this constraint. Also if M_f are considered large one can integrate out the ξ_f fermions this leading to an equivalent Lagrangian expressed only in terms of the light degrees of freedom Ψ_f.

5. DISCUSSION

When one constructs an effective theory for the low energy degrees of freedom of QCD, be these fermions or hadrons, one uses often as a guiding principle the approximate symmetries already established experimentally such as the chiral $SU(3)_L \times SU(3)_R$ symmetry or the $U(1)_A$ axial one. In the present work we considered a different approach; thus instead of dwelling on symmetries we focused on the intrinsic dynamics encapsulated in the partition function of the QCD Lagrangian. Using alternative sets of variables we were able to reproduce the exact partition function obtained after integrating out the quark degrees of freedom. It turns out that a partition function depending on a set of fermion variables can be described alternatively only by new sets of fermion variables, consider them copies of the first ones, that however have a completely different Lagrangian.

The final result (see Eq. (50)) is an effective Lagrangian with unusual terms depending on two groups of fermions (where Ψ_f are the three light quarks and ξ_f are the three heavy ones) and with couplings which go up to 8 fermion interaction terms. This Lagrangian gives an adequate partition function corresponding to that of the original QCD Lagrangian for the case when the coupling constant g is large. Moreover it has an intrinsic $SU(3)_L \times SU(3)_R$ chiral symmetry in the two sets of fermion sectors for the case when the quark masses m_f and M_f are set to zero and also an $SU(3)_V$ symmetry for the case when the two sets of masses are equal within one set.

The Lagrangian in Eq. (50) is of the same type as the Nambu Jona-Lasinio model [33], [34] which in its various versions has achieved remarkable success in describing not only the mass spectrum of the low lying scalars, pseudoscalars, vectors and axial vectors but also their main decaying modes. Both our model and the NJL one are nonrenormalizable models which enclose multiple fermion interactions but there are also important differences. Our Lagrangian incorporates four, six and eight quark interaction terms whereas the NJL model stops at four quark terms. The four fermion interaction term contains derivatives that however can be reduced by using adequate techniques. Moreover the Lagrangian in Eq. (50) includes heavy quark states that can be integrated out leading to a more comprehensive effective Lagrangian in terms of the light quark states.

The purely quark Lagrangain we obtained can be further processed by using QCD rules to extract the bound states of mesons or baryons and their interactions. However the overall method goes far beyond this. One could at any intermediate stage introduce directly instead of currents the meson states. In this case one must take into account that initially there are 72 degrees of freedom for the light quarks corresponding to a product of 3 colors, 3 flavors and 8 space time coordinates of an off-shell fermion. This means that the partition function must accommodate only that number of mesons, be they scalars, pseudoscalar, vectors etc. whose total number of degrees of freedom sum up 72. All alternatives that conveniently express the initial Lagrangian should be considered and all Lagrangians that can be constructed in this way are possible outcomes. This is also true for the Lagrangian in Eq. (50) as it is only one of the many possible choices compatible with the initial QCD partition function.

In this paper we propose a new theoretical effective fermion Lagrangian obtained from QCD not by integrating out the gluons but by eliminating these as a result of mocking up the exact partition function of QCD. We shall leave the discussion of phenomenological implications of our model for further work.

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