

QUASI-EXACTLY SOLVABLE DOUBLE-WELL POTENTIAL AND POLYNOMIAL DEFORMATIONS OF $sl(2)$ LIE ALGEBRA

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Abstract. Solutions of the Schrödinger equation with a quasi-exactly solvable double-well potential are investigated using the Lie algebraic approach. The algebraic structure of the corresponding equation is shown to be related with the deformations of $sl(2)$ Lie algebra. The corresponding generators are constructed through linear differential operators admitting a finite-dimensional subspace of monomials. With the help of the representation theory of $sl(2)$, exact expressions for the energies as well as the corresponding wave functions are obtained within the framework of quasi-exact solvability. We show that the method provides the exact results.

Key words: Quasi-exactly solvable double-well potential, deformations of $sl(2)$ Lie algebra, representation theory.

1. INTRODUCTION

Since the appearance of quantum mechanics, extensive efforts have been devoted to the determining the exact solutions of the wave equations, via different methods [1-8]. A quantum system is exactly solvable (ES) if all the eigenvalues and eigenfunctions can be found by algebraic methods. It is well known that ES models play an important role in various fields of physics and mathematics, but in general, except for a few cases, the Schrödinger equation cannot be solved exactly. During the last decades, an intermediate class between the ES models and the non-solvable models was introduced for which a finite part of the spectrum can be determined by purely algebraic means. For this reason, they were named quasi-exactly solvable (QES) models [9-14]. The idea of QES models has received a great deal of attention, both from the viewpoint of physical applications as well as from that of its inner mathematical beauty [15-20]. The QES models are distinguished by the fact that their Hamiltonians are expressed in terms of the generators of a finite-dimensional Lie algebra that possesses a finite-dimensional

representation space. As a result, the Hamiltonian reduces to a block diagonal matrix with at least one finite block. The eigenvalues and eigenfunctions corresponding to this block can be always calculated algebraically. A complete classification of the one-dimensional QES systems associated with the Lie algebra $sl(2)$ has been given by Turbiner [9,10]. The interested reader is referred to Refs. [21-25] for more information on the application of Lie group symmetries and efficient methods to solve linear and nonlinear partial differential equations. Recently, Roy *et al.* have shown that the quasi-exact solvability of the Heun equation (HE) and its confluent forms, can be related with the deformations of the $sl(2)$ Lie algebra [26]. The HE is the most general Fuchsian equation of second order with four regular singularities [27], and it is therefore a generalization of the hypergeometric equation (Fuchsian equation with three regular singularities). In recent years, HE and its confluent forms have received a considerable amount of attention due to a large number of their applications in quantum mechanics and mathematical physics, see [28-30] and references therein. On the other hand, it has been shown that several types of double-well potentials (DWPs) are related to the HE and its confluent forms. DWP is one of the fundamental problems in modern quantum mechanics [31]. Typical DWPs in the literature include the sextic oscillator potential [32, 33], quartic potential [34], generalized quantum isotonic oscillator [35], Razavy potential [36-38], and the Manning potential [39-41]. In this paper, we consider the following one-dimensional QES DWP

$$V(x) = \frac{V_1}{\cosh^2 x} + \frac{V_2}{1+g \cosh^2 x} + \frac{V_3}{(1+g \cosh^2 x)^2}, \quad (1)$$

which for $g \gg 1$, can be regarded as an approximation of the Manning potential [39]

$$V(x) = \frac{1}{\cosh^2 x} \left(V_1 + \frac{V_2}{g} \right) + \frac{1}{\cosh^4 x} \left(\frac{V_3}{g^2} \right). \quad (2)$$

Making a change of variable and an appropriate transformation, the Schrödinger equation for this DWP can be reduced to the HE. Chen *et al.* in Ref. [40] have obtained the solutions of this equation in terms of the Heun functions and also applied the Wronskian method to derive the conditions for the energy eigenvalues of the bound states. Here, we intend to study this equation via the polynomial deformations of the $sl(2)$ Lie algebra and then obtain the exact solutions in the representation space of $sl(2)$.

This paper is organized as follows. In Sec. 2, based on Ref. [26], we briefly review the connection between the polynomial deformations of $sl(2)$ Lie algebra and a class of differential equations with three regular singularities, of which HE is a particular case. Section 3 is devoted to the algebraization of the Schrödinger

equation with the QES DWP. We demonstrate that the system satisfy the cubic deformation of the $sl(2)$ algebraic structure, which is responsible for solvability of the problem. Also, using the representation theory of $sl(2)$, the exact expressions for the energies and the corresponding wave functions are obtained within the framework of quasi-exact solvability. We end with conclusions in Sec. 4.

2. HEUN CLASS OF DIFFERENTIAL EQUATIONS AND DEFORMED $sl(2)$ LIE ALGEBRA

According to Ref. [26], by making an appropriate change of variable and canonical transformation, the one-dimensional Schrödinger equation can be transformed to the following differential equation ($x \in \mathbb{R}$ or $x \in \mathbb{R} \geq 0$):

$$\left(h_1(x) \frac{d^2}{dx^2} + h_2(x) \frac{d}{dx} + h_3(x) \right) \psi(x) = 0, \quad (3)$$

where

$$\begin{aligned} h_1(x) &= a_0 x^3 + a_1 x^2 + a_2 x + a_3, \\ h_2(x) &= a_4 x^2 + a_5 x + a_6, \\ h_3(x) &= a_7 x + a_8, \end{aligned} \quad (4)$$

and a_i are real parameters for $i = 1, 2, \dots, 8$. For different values of a_i , eq. (3) is transformed to a wide class of differential equations, including Heun, confluent Heun, bi-confluent Heun, doubly confluent Heun, and Jacobi. Let us first recall the Lie algebraic approach of quasi-exact solvability introduced in Ref. [9]. A linear differential equation T on a Hilbert space \mathbf{H} is called QES if it leaves invariant a finite-dimensional subspace $P \subset \mathbf{H}$, i.e.

$$\begin{aligned} TP &\subset P, \\ P &= \langle \tau_1, \tau_2, \dots, \tau_n \rangle, \quad \tau_i \in \mathbf{H}. \end{aligned} \quad (5)$$

Hence, the first n eigenvalues and eigenfunctions can be determined exactly. In one dimension, the only Lie algebra of first-order differential operators which possesses finite-dimensional representations is $sl(2)$, whose generators

$$\begin{aligned}
 J^+ &= x^2 \frac{d}{dx} - 2jx, \\
 J^0 &= x \frac{d}{dx} - j, \\
 J^- &= \frac{d}{dx}, \quad j = 0, \frac{1}{2}, 1, \dots,
 \end{aligned} \tag{6}$$

satisfy the commutation relations

$$[J^+, J^-] = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm, \tag{7}$$

and leave invariant the $(2j+1)$ -dimensional space of monomials x^{j+m} ($m \leq |j|$). The interested reader is referred to Refs. [9,10] for a complete introduction to the QES differential equations related to the $sl(2)$ algebra. Recently, Roy *et al.* in Ref. [26] have shown that the symmetry of the QES problems for the general differential equation (3), can be related with the deformations of the $sl(2)$ Lie algebra with a set of operators such that

$$P_+ x^{2j} = C_+ x^{2j+1}, \quad P_0 x^{2j} = C_0 x^{2j}, \quad P_- x^{2j} = C_- x^{2j-1}. \tag{8}$$

Then, if $a_3 = 0$, eq. (3) is expressible as a linear combinations of operator P_+ , P_0 , and P_- as:

$$(P_+ + F(P_0) + P_-)\psi(x) = 0, \tag{9}$$

where the operators

$$\begin{aligned}
 P_+ &= a_0 x^3 \frac{d^2}{dx^2} + a_4 x^2 \frac{d}{dx} + a_7 x, \\
 P_0 &= x \frac{d}{dx} - j, \quad F(x \frac{d}{dx}) = a_1 x^2 \frac{d^2}{dx^2} + a_5 x \frac{d}{dx} + a_8, \\
 P_- &= a_2 x \frac{d^2}{dx^2} + a_6 \frac{d}{dx},
 \end{aligned} \tag{10}$$

generate the following deformation of $sl(2)$ Lie algebra

$$\begin{aligned}
 [P_0, P_\pm] &= \pm P_\pm, \\
 [P_+, P_-] &= \alpha_1 P_0^3 + \alpha_2 P_0^2 + \alpha_3 P_0 + \alpha_4 = f(P_0).
 \end{aligned} \tag{11}$$

Here, α_i ($i = 1, 2, 3, 4$) are real parameters in terms of a_i 's and j [26]. Also, the Casimir operator associated with this algebra is $C = J^- J^+ + g(j)$ with the eigenvalue $a_6 a_7$ and $g(j) - g(j-1) = f(j)$ [42]. In the next section, we apply the above method to obtain the exact solutions of the QES DWP. We start by reviewing the work of Chen *et al.*, who analytically studied the problem via the Heun functions [40].

3. DOUBLE-WELL POTENTIAL AND DEFORMED $sl(2)$ ALGEBRA

The Schrödinger equation with the QES DWP (1) is written as ($\hbar = 2m = 1$) [40]

$$\left(-\frac{d^2}{dx^2} + \frac{V_1}{\cosh^2 x} + \frac{V_2}{1 + g \cosh^2 x} + \frac{V_3}{(1 + g \cosh^2 x)^2} \right) \psi(x) = E \psi(x). \quad (12)$$

Making the change of variable $z = -\sinh^2 x$ and also by considering the following transformation

$$\psi(x) = (\cosh(x))^{2\lambda_1} (1 + g \cosh^2(x))^{\lambda_2} \varphi(z), \quad (13)$$

where the parameters λ_1 and λ_2 are determined from the asymptotic behavior of the wave function as

$$\lambda_1 = \frac{1 + \sqrt{1 - 4V_1}}{4}, \quad \lambda_2 = \frac{1 - \sqrt{1 + V_3 / (1 + g)}}{2}, \quad (14)$$

eq. (12) transforms into the following Heun differential equation

$$\frac{d^2 \varphi}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) \frac{d\varphi}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} \varphi = 0, \quad (15)$$

where

$$\begin{aligned} \gamma &= \frac{1}{2}, & \delta &= 2\lambda_1 + \frac{1}{2}, & a &= \frac{1+g}{g}, \\ \alpha &= \lambda_1 + \lambda_2 + \frac{\sqrt{E}}{2}, & \beta &= \lambda_1 + \lambda_2 - \frac{\sqrt{E}}{2}, \\ q &= \left(2\lambda_1 + 2\frac{\lambda_2}{a} - V_1 - \frac{V_2}{ga} - \frac{V_3}{(ga)^2} + E \right) \frac{a}{4}, \\ \varepsilon &= \alpha + \beta + 1 - \gamma - \delta = 2\lambda_2. \end{aligned} \quad (16)$$

Solutions of this equation have been analytically obtained in terms of the Heun functions in Ref. [40]. Here, we intend to study this equation via the polynomial deformations of the $sl(2)$ Lie algebra which enables us to obtain the exact solutions using the representation theory of the $sl(2)$ Lie algebra. In order to relate the eq. (15) to the polynomial deformations of the $sl(2)$ Lie algebra, it is convenient to write it in the following form

$$\left\{ P_3(z) \frac{d^2}{dz^2} + P_2(z) \frac{d}{dz} + P_1(z) \right\} \varphi(z) = 0, \quad (17)$$

where

$$\begin{aligned} P_3(z) &= z^3 - (a+1)z^2 + az, \\ P_2(z) &= (\gamma + \delta + \varepsilon)z^2 - (\gamma(a+1) + a\delta + \varepsilon)z + a\gamma, \\ P_1(z) &= \alpha\beta z - q. \end{aligned} \quad (18)$$

Comparing eq. (17) with eq. (9), the generators of the deformed $sl(2)$ algebra are obtained as

$$\begin{aligned} P_+ &= z^3 \frac{d^2}{dz^2} + (\gamma + \delta + \varepsilon)z^2 \frac{d}{dz} + \alpha\beta z, \\ P_- &= z \frac{d^2}{dz^2} + \gamma \frac{d}{dz}, \\ P_0 &= z \frac{d}{dz} - j, \quad F\left(z \frac{d}{dz}\right) = n_2 P_0^2 + n_1 P_0 + n_0, \end{aligned} \quad (19)$$

where

$$\begin{aligned} n_0 &= -(a+1)j^2 + (a - (\gamma(a+1) + a\delta + \varepsilon) + 1) - q, \\ n_1 &= (a - (\gamma(a+1) + a\delta + \varepsilon) + 1) - 2j(a+1), \\ n_2 &= -(a+1). \end{aligned} \quad (20)$$

Obviously, these generators satisfy the commutation relations (11). Therefore, eq. (15) is a QES differential equation related to the polynomial deformations of $sl(2)$ algebra and hence, preserves the finite-dimensional representation space of $sl(2)$ spanned by the basis $\langle 1, z, z^2, \dots, z^{2j} \rangle$. Considering the solutions of eq. (17) around the regular singular point $z=0$ as the polynomial solutions

$$\varphi(z) = \sum_{m=0}^{2j} C_m z^m, \quad (21)$$

and substituting in eq. (17), we obtain the following recursion relation for the expansion coefficients C_m

$$C_{m+1} = \frac{(m((m-1+\gamma)(1+a)+a\delta+\varepsilon)+q)C_m - (m-1+\alpha)(m-1+\beta)C_{m-1}}{(a(m+1)(m+\gamma))}, \quad (22)$$

with $C_{-1}=0$ and $C_{2j+1}=0$. According to the characterization of quasi-exact solvability, eq. (17) must preserve the space of polynomials in z of degree at most $2j$. It is easy to verify that

$$\left\{ P_3(z) \frac{d^2}{dz^2} + P_2(z) \frac{d}{dz} + P_1(z) \right\} z^{2j} = \quad (23)$$

$$(4j^2 - 2j + 2j(\gamma + \delta + \varepsilon) + \alpha\beta) r^{2j+1} + \text{lower order terms},$$

which implies that the key point in obtaining the exact solutions of the problem is the introduction of the nonnegative integer $n = 2j$ such that

$$n^2 - n + (\gamma + \delta + \varepsilon)n + \alpha\beta = 0. \quad (24)$$

Here, for comparison purpose, we take the parameters V_1 , V_2 , and V_3 as in Ref. [40]

$$\begin{aligned} V_1 &= g(g+2)/4(1+g)^2, \\ V_2 &= -4b^2(g+2), \\ V_3 &= -4b(g+1)(b+1). \end{aligned} \quad (25)$$

Therefore, substituting the parameters from (16) and (25) into (24), yields the closed expression for the energy of the problem as

$$E_n = -\frac{1}{4} \frac{(2+4n+g(n+1)-4b(g+1))^2}{(1+g)^2}. \quad (26)$$

In the following, we determine the ground state and the first excited state of the system. For $n=0$, from eqs. (26), (13), and (21), we obtain the following relations

$$E_0 = -\frac{(2+g-4b(1+g))^2}{4(1+g)^2}, \quad (27)$$

and

$$\psi_0(x) = N_0 (\cosh x)^{\frac{g+2}{2(1+g)}} (1+g \cosh^2 x)^{-b}, \quad (28)$$

for the energy and the corresponding wave function, respectively. For the first excited state, i.e., $n=1$, from Eqs. (26), (13) and (21), we obtain

$$E_1 = -\frac{(6+2g-4b(1+g))^2}{4(1+g)^2}, \quad (29)$$

$$\psi_1(x) = N_1 (\cosh x)^{\frac{g+2}{2(1+g)}} (1+g \cosh^2 x)^{-b} \left(1 - \frac{q}{a\gamma} \sinh^2 x\right). \quad (30)$$

Here N_0 and N_1 are the normalization constants. Similarly, the remaining part of the spectrum can be determined. Now, we consider another linearly independent series solution of eq. (17) around $z=0$ as

$$\tilde{\varphi}(z) = z^{1-\gamma} \sum_{m=0}^{2j} \tilde{C}_m z^m, \quad (31)$$

where the expansion coefficients \tilde{C}_m obey the three-term recursion relation

$$\tilde{C}_{m+1} = \frac{((m+1+\gamma)((m+1+a)+a\delta+\varepsilon)+q)\tilde{C}_m - (m+\alpha-\gamma)(m+\beta-\gamma)\tilde{C}_{m-1}}{a(m+2-\gamma)(m+1)}, \quad (32)$$

with $\tilde{C}_{-1}=0$ and $\tilde{C}_{2j+1}=0$. Following the same procedure as before, the condition of solvability is obtained as

$$(n+1-\gamma)(n-\gamma) + (\gamma+\delta+\varepsilon)(n+1-\gamma) + \alpha\beta = 0, \quad (33)$$

which by substituting the parameters from (16) and (25), the closed expression for energy of the system is obtained as

$$\tilde{E}_n = -\frac{1}{4} \frac{(4+4n+g(4n+3)-4b(g+1))^2}{(1+g)^2}. \quad (34)$$

For $n=0$, from eqs. (34), (13), and (31), we obtain

$$\tilde{E}_0 = -\frac{(4 + 3g - 4b(1 + g))^2}{4(1 + g)^2}, \quad (35)$$

$$\tilde{\psi}_0(x) = \tilde{N}_0 (\cosh x)^{\frac{g+2}{2(1+g)}} (1 + g \cosh^2 x)^{-b} \sinh x, \quad (36)$$

for energy and wave function, respectively. Likewise, for $n = 1$, from eqs. (34), (13), and (31), we obtain

$$\tilde{E}_1 = -\frac{(8 + 7g - 4b(1 + g))^2}{4(1 + g)^2}, \quad (37)$$

$$\tilde{\psi}_1(x) = \tilde{N}_1 (\cosh x)^{\frac{g+2}{2(1+g)}} (1 + g \cosh^2 x)^{-b} (1 - A \sinh^2 x) \sinh x, \quad (38)$$

where

$$A = \frac{((\gamma + 1)(a\delta + \varepsilon) + q)}{a(2 - \gamma)}. \quad (39)$$

Similarly, the other excited states can be easily obtained from eqs. (34), (13), and (31). Our results for the ground state and the first excited state are summarized in Table 1. It should be noted that our results are identical with the results obtained using the Heun functions in Ref. [40].

Table 1.

Exact solutions of the ground state and the first excited state.

n		$\psi(x)$
0	$E_0 = -\frac{(2 + g - 4b(1 + g))^2}{4(1 + g)^2}$ $\tilde{E}_0 = -\frac{(4 + 3g - 4b(1 + g))^2}{4(1 + g)^2}$	$\psi_0(x) = N_0 (\cosh x)^{\frac{g+2}{2(1+g)}} (1 + g \cosh^2 x)^{-b}$ $\tilde{\psi}_0(x) = \tilde{N}_0 (\cosh x)^{\frac{g+2}{2(1+g)}} (1 + g \cosh^2 x)^{-b} \sinh x$
1	$E_1 = -\frac{(6 + 2g - 4b(1 + g))^2}{4(1 + g)^2}$ $\tilde{E}_1 = -\frac{(8 + 7g - 4b(1 + g))^2}{4(1 + g)^2}$	$\psi_1(x) = N_1 (\cosh x)^{\frac{g+2}{2(1+g)}} (1 + g \cosh^2 x)^{-b} \left(1 - \frac{q}{a\gamma} \sinh^2 x\right)$ $\tilde{\psi}_1(x) = \tilde{N}_1 (\cosh x)^{\frac{g+2}{2(1+g)}} (1 + g \cosh^2 x)^{-b} (1 - A \sinh^2 x) \sinh x$

4. CONCLUSION

By using the Lie algebraic approach, we have investigated the algebraic solutions of the Schrödinger equation with a quasi-exactly solvable double-well potential. We have shown that the corresponding equation is reducible to a QES differential equation with a hidden algebraic structure related to the polynomial deformations of $sl(2)$. Using the representation theory, we have obtained exact expressions for the energies as well as wave functions within the framework of quasi-exact solvability. We have also shown that our results are identical with those obtained by other methods.

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