COMPOSITE SOLITONS IN TWO-DIMENSIONAL SPIN-ORBIT COUPLED SELF-ATTRACTIVE BOSE-EINSTEIN CONDENSATES IN FREE SPACE

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Abstract. We review properties of two-dimensional matter-wave solitons, governed by the spinor system of Gross-Pitaevskii equations with cubic nonlinearity, including spin-orbit coupling and the Zeeman splitting. In contrast to the collapse instability typical for the free space, spin-orbit coupling gives rise to stable solitary vortices. These are semi-vortices with a vortex in one spin component and a fundamental soliton in the other, and mixed modes, with topological charges 0 and ±1 present in both components. The semivortices and mixed modes realize the ground state of the system, provided that the self-attraction in the spinor components is, respectively, stronger or weaker than their cross-attraction. The modes of both types degenerate into unstable Townes solitons when their norms attain the respective critical values, while there is no lower norm threshold for the stable modes existence. With the Galilean invariance lifted by the spin-orbit coupling, moving stable solitons can exist up to a mode-dependent critical velocity with two moving solitons merging into a single one as a result of collision. Augmenting the Rashba term by the Dresselhaus coupling has a destructive effect on these states. The Zeeman splitting tends to convert the mixed modes into the semivortices, which eventually suffer delocalization. Existence domains for the soliton families are reviewed in terms of experiment-related quantities.

Key words: two-dimensional matter-wave solitons, spin-orbit coupling, Zeeman splitting.

1. INTRODUCTION: SYNTHETIC SPIN-ORBIT COUPLING AND NONLINEARITIES

Macroscopic cold atomic systems have become testbeds for many effects in condensed matter physics [1]. Bosonic gases, optically cooled down to the ultralow temperature of the Bose-Einstein condensation (BEC) [2–4], allow the researchers to study a broad variety of phenomena, including effects of interatomic interaction, i.e., various forms of effective nonlinearities common for condensed matter [5] and
Although the true spin of bosonic atoms, such as, e.g., $^{87}$Rb, used in the BEC related experiments, is zero, the wave function of the condensate in a highly coherent resonant optical field has two components related to the hyperfine splitting of the atomic levels. As a result, physics of pseudospin $1/2$ emerges, similar to that of electrons in solids. More recently, a great attention has been drawn to the experimentally demonstrated [7] ability to emulate the spin-orbit coupling (SOC) in the form of a linear interaction between the atomic momentum and pseudospin, in the combination with the Zeeman-splitting (ZS) effect [8]. Both fundamental types of the SOC, represented by the Dresselhaus [9] and Rashba [10] Hamiltonians, can be simulated in the cold-matter systems. A majority of experimental works on the SOC [11] were dealing with effectively one-dimensional SOC settings. Recently, the realization of the SOC in the two-dimensional (2D) BEC was reported [12], suggesting a deeper understanding of possible 2D effects. A similar method is also used to produce the orbital gauge fields [13, 14].

The interplay of the linear SOC and the BEC nonlinearity produces many nontrivial localized structures: vortices [15–18], monopoles [19], skyrmions [20, 21], solitons [22, 23], etc. A comprehensive review is presented in Ref. [24]. Trapping of condensates in 2D potentials [18] produces vortices similar to those found in three-dimensional (3D) settings with spatially modulated self-repulsion [25]. Optical lattices open new realizations of the SOC [26], such as the creation of gap solitons [27]. The similarity between the Gross-Pitaevskii equations (GPEs) for the SOC binary condensate [22] and the model of the co-propagation of orthogonal polarizations of light in twisted nonlinear optical fibers [28] puts the SOC in the broad context of nonlinear phenomena. The strong linear coupling of pseudospin $1/2$ to the atomic momentum connects the physics of the spinor BEC to graphene physics and its emulation in photonic crystals [29].

In addition to the conventional repulsive interatomic forces, attractive interactions can also take place in BEC, by means of the Feshbach resonance [30–32], which is realizable in most known alkali-metal- [33–37], rare-earth- [38, 39], and transition metal-based [40] atomic BEC species. It is natural to expect the existence of solitons in condensates with the self-attractive intrinsic nonlinearity. In particular, while the concept of 2D solitons with embedded vorticity, introduced in Ref. [41], is well known, in the free space such states are usually subject to strong splitting instabilities, even if their zero-vorticity counterparts (fundamental solitons) do not suffer collapse [42]. A theoretical challenge in studies of solitons is related to the fact that the Thomas-Fermi approximation, which offers a powerful analytical tool valid for the mean-field models with self-repulsion [18, 25], does not apply to the case of attraction. Using, instead, numerical methods, it has been found, in particular, that trapping potentials can stabilize vortex solitons against the collapse and
splitting alike, under the action of various interactions [43–47]. Further, it is expected that both single vortices and vortex lattices will be nontrivially affected if the SOC is present in self-attractive condensates [15, 16].

Here we review the results obtained in recent studies of the mean-field models for self-attractive condensates under the action of SOC in the free space without the use of trapping potentials. We demonstrate a surprising result that two different families of vortex solitons, namely semi-vortices (SVs, with topological charges \( m = 0 \) and \( \pm 1 \) separately carried by the two components) and mixed modes (MMs, which combine \( m = (0, -1) \) and \( m = (0, +1) \) in the components) becomes stable in the 2D binary system with the linear SOC of the Rashba type. This result was reported in Ref. [46], and further extended in Refs. [48, 49]. The SV (MM) solitons realize the ground state of the 2D system when the spinor-diagonal self-attraction is stronger (weaker) than its off-diagonal counterpart. These spinor-diagonal and off-diagonal interactions are similar to the self-phase-modulation and cross-phase-modulation interactions, respectively, in nonlinear optics [50]. It is relevant to mention too that, in other contexts (chiefly, in models of nonlinear optics), 2D solitons can be stabilized by nonlocal self-attraction [51].

In the case of self-attraction, a commonly known problem is that 2D zero-vorticity solitons, supported by cubic terms, are strongly unstable in the free space, due to the occurrence of the critical wave collapse [52], while vortical solitons are subject to a still stronger splitting instability [53]. In particular, in the 2D case, the Gross-Pitaevskii (alias nonlinear Schrödinger) equation with the cubic self-attraction term gives rise to degenerate families of fundamental Townes solitons with \( m = 0 \) [54] and their vortical counterparts with \( m \geq 1 \) [41]. The degeneracy means that the entire family share a single value of the norm, which is, as a matter of fact, one separating collapsing and decaying solutions. Hence the Townes solitons, that play the role of separatrices between these two types of the dynamical behavior, are themselves completely unstable. In turn, the degeneracy is a consequence of the specific scale invariance of the cubic GPE in two dimensions.

The explanation to the stabilizing effect of the SOC for the 2D solitons in free space is that SOC is characterized by an additional parameter (as shown in some detail below), viz., the spin-precession length [13, 14], which is inversely proportional to the SOC strength. This fixed length scale lifts the above-mentioned scale invariance, lifting the norm degeneracy of the solitons, and pushing their norm below the threshold necessary for the onset of the critical collapse. Being protected against the collapse, the solitons enjoy stabilization and actually introduce a ground state, which is missing in the scale-invariant 2D systems with the cubic self-attraction [46, 48, 49].

Theaptitude of the SOC terms in 2D to avoid the critical collapse was also demonstrated, in other contexts, in Refs. [55] and [56]. Definitely, the stable free-space 2D solitons are of interest to many other nonlinear systems [57, 58].
Making a relation to the 3D systems, where the cubic self-attraction gives rise to supercritical collapse [52], we note that, although the SOC cannot suppress the collapse [59], the interplay of the linear SOC and cubic self-attraction forms metastable solitons of the same two generic types, SV and MM [59]. Being metastable, these states are robust against weak perturbations but can collapse as a result of a sudden strong compression driving them to the instability domain.

Because a physically relevant generic situation includes a combination of the SOC of the Rashba and Dresselhaus types, here we review the physics of the 2D solitons governed by the full Rashba-Dresselhaus Hamiltonian too, following our work [47]. The Dresselhaus interaction reduces the solitons’ stability regions and produces the stability boundaries. Eventually, the Dresselhaus coupling leads to delocalization of 2D solitons. To complete the picture, we review the effects of the Zeeman splitting and show that the increase in the Zeeman term causes a transition from the MM states to the SV ones and a very strong Zeeman field destroys the solitons. We study all reviewed realizations with combined analytical and numerical approaches, where the variational approximation serves as a reliable tool for the qualitative and semi-quantitative analysis.

2. THE MODEL AND BASIC EQUATIONS

We begin with the general consideration of the effects of the synthetic SOC (including both the Rashba and the Dresselhaus terms) and the Zeeman splitting, in 2D space \((x, y)\). In the mean-field approximation [3, 4, 60], the pseudo-spinor condensate is described by a two-component wave function [13, 14],

\[
\phi^+, \phi^-
\]

transposed, with total norm

\[
|N| = \int \int (|\phi^+|^2 + |\phi^-|^2) dx dy \equiv N^+ + N^-,
\]

which is proportional to the total number of atoms in the condensate. The applicability of this approach to SOC settings was demonstrated in many works [13, 14], including the case of attractive interactions [61]. Accordingly, the evolution of the wave function is governed by the system of coupled GPEs, written here in the scaled form [13, 14]:

\[
\begin{align*}
\frac{i}{\hbar} \partial \phi^+ &= -\frac{1}{2} \nabla^2 \phi^+ - \left( |\phi^+|^2 + \gamma |\phi^-|^2 \right) \phi^+ + \left( \lambda \tilde{D}^\dagger \phi^- - i \lambda_D \tilde{D}^\dagger \phi^- \right) - \Omega \phi^+, \\
\frac{i}{\hbar} \partial \phi^- &= -\frac{1}{2} \nabla^2 \phi^- - \left( |\phi^-|^2 + \gamma |\phi^+|^2 \right) \phi^- - \left( \lambda \tilde{D}^\dagger \phi^+ + i \lambda_D \tilde{D}^\dagger \phi^+ \right) + \Omega \phi^-,
\end{align*}
\]

where \(\lambda\) and \(\lambda_D\) are constants of the Rashba and Dresselhaus coupling, respectively, the nonlinear interactions are assumed to be attractive, \(\gamma\) being the relative strength of the cross-attraction between the two components, while the self-attraction strength is scaled to be 1, and we introduced operators \(\tilde{D}^{[\pm]} \equiv \partial / \partial x \pm i \partial / \partial y\). The ZS term
with strength $\Omega$ corresponds to the synthetic magnetic field directed along the $z$–axis [11], lifting the time-reversal symmetry between the components of the spinor wave function.

The comparison of scaled 2D equations (2) and (3) with the underlying system of 3D GPEs, written in the physical units, shows that the unit length in these equations corresponds to the spatial scale $\sim 1 \mu m$. Further, by assuming typical values of the transverse confinement length $\sim 3 \mu m$ and the scattering length of the interatomic attraction $\sim -0.1 \text{ nm}$, we find that $N = 1$ in the present notation is tantamount to $\sim 3 \times 10^3$ atoms.

The spectrum of plane waves generated by the linearized version of Eqs. (2), (3), $\phi_\pm \sim \exp \left( i k \cdot r - i \epsilon_\pm t \right)$, where $k$ is the wave vector, contains two branches:

$$\epsilon_\pm = \frac{k^2}{2} \pm \sqrt{(\lambda^2 + \lambda_D^2)k^2 + 4\lambda\lambda_D k_x k_y + \Omega^2},$$

with a quasi-gap $2\Omega$ at $k = 0$ [58]. In terms of the estimate for physical parameters given above, a characteristic strength $\Omega = 1$ corresponds, in physical units, to $\sim 2\pi \times 100 \text{ Hz}$ for $^{85}\text{Rb}$, or $2\pi \times 1 \text{ KHz}$ for $^7\text{Li}$. This spectrum is anisotropic and at $\lambda = \pm \lambda_D$ the spin-dependent term becomes effectively one-dimensional, as it has originally been realized in the BEC settings [7]. This anisotropy qualitatively alters the solitons, and eventually causes their delocalization, as shown below.

There are two qualitative general features of the SOC important for the understanding of the nonlinear systems. For clarity, we set $\lambda_D = 0$ and $\Omega = 0$, while discussing these features.

1) The SOC leads to the spin precession with the rate $\omega = 2\lambda k$, and the respective spin-precession angle $\varphi = \omega t = \frac{2L}{L_{soc}}$, where $L$ is the particle displacement, and the precession length is $L_{soc} \equiv \frac{1}{\lambda}$. For typical realizations [11, 12], the value of $L_{soc}$, is of the order of few micron. This length establishes a spatial scale in the setting and manifests itself in the formation of solitons.

2) The SOC lifts the Galilean invariance, as can be seen from the appearance of the spin-dependent anomalous velocity proportional to the SOC strength (see, e.g., Ref. [55]). Indeed, by calculating quantum-mechanical expectation values of velocities corresponding to Eqs. (2) and (3), we obtain:

$$\langle v_x \rangle_\phi \equiv \frac{\partial \langle x \rangle_\phi}{\partial t} = \langle \hat{k}_x \rangle_\phi + \lambda \langle \sigma_y \rangle_\phi; \quad \langle v_y \rangle_\phi \equiv \frac{\partial \langle y \rangle_\phi}{\partial t} = \langle \hat{k}_y \rangle_\phi - \lambda \langle \sigma_x \rangle_\phi,$$

where the momentum operator components are given by $\hat{k}_x \equiv -i\partial / \partial x$ and $\hat{k}_y \equiv -i\partial / \partial y$, respectively, and $\langle \cdots \rangle_\phi$ stands for expectation values of operators at given spinor state $\phi$. Therefore, the system can exhibit motion even when $\langle \hat{k}_x \rangle_\phi = \langle \hat{k}_y \rangle_\phi = 0$ provided that at least one of the expectation values of the spin components ex-
pressed as
\[
\langle \sigma_x \rangle_\phi = \frac{2}{N} \iint \text{Re} (\phi^*_+ \phi_-) \, dx \, dy, \quad \langle \sigma_y \rangle_\phi = \frac{2}{N} \iint \text{Im} (\phi^*_+ \phi_-) \, dx \, dy,
\]  
(6)
is not zero. This has consequences for the dynamics of the vortex solitons, producing moving modes up to a certain critical velocity.

The exact equations are often difficult to solve directly, suggesting one to employ the variational approach, based on the minimization of the system’s energy at a fixed value of the norm. The total energy of the system is the sum of the kinetic \((E_k)\), self-interaction \((E_{\text{int}})\), SOC \((E_{\text{soc}})\), and ZS \((E_Z)\) terms:
\[
E = E_k + E_{\text{int}} + E_{\text{soc}} + E_Z,
\]
(7)
where
\[
E_k = \frac{1}{2} \iint (|\nabla \phi_+|^2 + |\nabla \phi_-|^2) \, dx \, dy,
\]
(8)
\[
E_{\text{int}} = -\frac{1}{2} \iint \left[ (|\phi_+|^4 + |\phi_-|^4) + 2\gamma |\phi_+|^2 |\phi_-|^2 \right] \, dx \, dy,
\]
(9)
\[
E_{\text{soc}} = \iint \left[ \phi^*_+ \left( \lambda \hat{D}^[-] - i\lambda_D \hat{D}^{[+] } \right) \phi_- - \phi^*_- \left( \lambda \hat{D}^{[+] } + i\lambda_D \hat{D}^[-] \right) \phi_+ \right] \, dx \, dy,
\]
(10)
\[
E_Z = -\Omega \iint (|\phi_+|^2 - |\phi_-|^2) \, dx \, dy.
\]
(11)

This representation shows the importance of the SOC and the ZS terms: the spin-dependent energies \(E_{\text{soc}}\) and \(E_Z\) affect the spinor components \(\phi_+\) and \(\phi_-\), and, therefore, the \(E_{\text{int}}\) and \(E_k\) contributions too. Thus, the ground state is determined by a nontrivial interplay of all the energy terms.

3. SOLITARY VORTICAL MODES WITH THE RASHBA COUPLING

To understand the appearance of novel types of 2D solitons, we begin with the basic model combining the cubic self-attraction and the SOC terms of the Rashba type, in the absence of the Zeeman term [46].

3.1. SEMI-VORTEX SOLITONS

First, we note that Eqs. (2) and (3) admit stationary solutions as bound states of a fundamental soliton in component \(\phi_+\) (with zero vorticity, \(m_+ = 0\)), and a solitary vortex, with \(m_- = 1\), in \(\phi_-\):
\[
\phi_+ (x, y, t) = e^{-i\mu t} f_1 (r^2), \quad \phi_- (x, y, t) = e^{-i\mu t + i\theta} r f_2 (r^2),
\]
(12)
where \( \mu \) is the chemical potential and \((r, \theta)\) are the polar coordinates in the \((x, y)\) plane. Accordingly, composite modes of this type may be called semi-vortices (SVs; similar delocalized composite modes, found in Ref. [17] in a model with the repulsive nonlinearity, were called “half vortices”). The functions \( f_{1,2}(r^2) \) from ansatz (12) obey the following equations:

\[
\begin{align*}
\mu f_1 + 2 \left[ r^2 \frac{d^2 f_1}{d(r^2)^2} + \frac{d f_1}{d(r^2)} \right] + \left( f_1^2 + \gamma r^2 f_2^2 \right) f_1 - 2\lambda \left( r^2 f_2' + f_2 \right) &= 0, \\
\mu f_2 + 2 \left[ r^2 \frac{d^2 f_2}{d(r^2)^2} + 2 \frac{d f_2}{d(r^2)} \right] + \left( r^2 f_2^2 + \gamma f_1^2 \right) f_2 + 2\lambda f_1' &= 0. 
\end{align*}
\]

The invariance of Eqs. (13) with respect to the transformation

\[
\phi_{\pm}(r, \theta) \to \phi_{\mp}(r, \pi - \theta),
\]

gives rise to a SV that is a mirror image of (12), with the vorticities \((m_+, m_-) = (0, 1)\) replaced by time-reversed \((m_+, m_-) = (-1, 0)\):

\[
\phi_+(x, y, t) = -e^{-i\mu t} e^{-i\theta} e^{r/R_{SV}} \sqrt{r} \cos \left( \frac{r}{L_{soc}} + \delta \right), \quad \phi_- = e^{-i\mu t} f_1(r^2).
\]

An asymptotic form of the SV solution to Eqs. (2), (3) at \( r \to \infty \) was found in Ref. [46]:

\[
\begin{align*}
\phi_+(x, y, t) &\approx F e^{-i\mu t} e^{-r/R_{SV}} \sqrt{r} \cos \left( \frac{r}{L_{soc}} + \delta \right), \\
\phi_-(x, y, t) &\approx -F e^{-i\mu t + i\theta} e^{-r/R_{SV}} \sqrt{r} \sin \left( \frac{r}{L_{soc}} + \delta \right),
\end{align*}
\]

where \( F \) and \( \delta \) are arbitrary real constants, \( R_{SV} = 1/\sqrt{-2\mu + \lambda^2} \) is the localization radius, and \( L_{soc} = 1/\lambda \) is the above defined spin-precession length, which effectively determines the SV spatial scale. Thus, the localized modes exist at values of the chemical potential

\[
\mu < -\lambda^2/2,
\]

belonging to the semi-infinite gap in the spectrum given by Eq. (4), and \( R_{SV} \) diverges when \( \mu \) is approaching \(-\lambda^2/2\). In fact, Eq. (16) gives the asymptotic form of the free-space solitons not only for the SVs, but in the general case too. For negative \( \mu \) with \(|\mu| \gg \lambda^2\), one obtains \( R_{SV} \ll L_{soc} \). Therefore, the spin-orbit coupling becomes inefficient, and stable soliton should disappear.

As shown below, the free-space SVs represent the ground state of the system at \( \gamma \leq 1 \). The coexistence of the two species of mutually-symmetric vortices (12) and (15) implies the degeneracy of the ground state, which is possible in nonlinear quantum systems, while prohibited in linear ones.

Stable SVs were first generated, as solutions to Eqs. (2),(3) with \( \gamma = 0 \) (no cross-nonlinearity) by means of imaginary-time simulations [62, 63], starting with
input
\[
\left( \phi_+^0 \right)_{SV} = A_1 e^{-\alpha_1 r^2}, \quad \left( \phi_-^0 \right)_{SV} = A_2 r e^{i\theta - \alpha_2 r^2},
\]
where \( A_{1,2} \) and \( \alpha_{1,2} > 0 \) are real constants. Obviously, this input conforms to the general SV ansatz (12). To emulate an experimentally feasible scenario, the solution was originally constructed in this way in the presence of the harmonic oscillator potential, which was then adiabatically switched off in real time, producing a stable state in the free space. The same SV, shown in Fig. 1(a), can be found directly in the free space by means of the same imaginary-time integration.

Further, Fig. 1(b) shows the dependence of the chemical potential \( \mu \) for the SV states on the state norm \( N \). The decreasing \( \mu(N) \) dependence satisfies the Vakhitov-Kolokolov criterion [64, 65], which is a necessary, but not a sufficient condition for the stability of solitons supported by the self-attraction. We stress that, as Fig. 1(b) demonstrates, the free-space SVs exist up to \( N \to 0 \), such that there is no threshold norm form their existence.

The spinor wave form (18) can be used not only as the input for the imaginary-time simulations, but also as a variational ansatz for the description of the SVs. Its substitution in the expression for the total energy corresponding to Eq. (7) yields
\[
E_{SV} = \pi \left( \frac{A_1^2}{2} - \frac{A_1^4}{8\alpha_1} + \frac{A_2^2}{2\alpha_2} - \frac{A_2^4}{64\alpha_2^2} \right) - \frac{\gamma A_1^2 A_2^2}{4(\alpha_1 + \alpha_2)^2} + \frac{4\lambda A_1 A_2 \alpha_1}{(\alpha_1 + \alpha_2)}),
\]
while the total norm (1) is \( N = \pi \left( A_1^2/2\alpha_1 + A_2^2/4\alpha_2^2 \right) \). Then, amplitudes \( A_1, A_2 \) and inverse squared widths \( \alpha_1 \) and \( \alpha_2 \) of the ansatz are obtained by the conditional minimization of total energy with respect to the variational parameters, \( \partial(E_{SV} - \mu N)/\partial(A_{1,2}, \alpha_{1,2}) = 0 \) with \( \mu \) being the corresponding Lagrange multiplier introduced for keeping \( N \) constant. The SV family exists at \( N < N_c \approx 5.85 \), the latter value being the well-known collapse threshold, i.e., the norm of the fundamental 2D Townes solitons [65]. Indeed, Fig. 1(c) shows, by means of the dependence of ratio

![Fig. 1 – Structure of the semi-vortices. (a) Profiles of the free-space semi-vortex \( |\phi_+ (x, 0)| \) and \( |\phi_- (x, 0)| \) (as marked near the lines). Here \( N = 5 \), \( \lambda = 1 \), and \( \gamma = 0 \). (b) Chemical potential \( \mu \) vs. norm \( N \) for the family of localized semi-vortices. (c) Ratio \( N_+ / N \) as a function of \( N \).](image-url)
$N_+/N$ on $N$, that the vortical component $\phi_-$ vanishes at $N \to N_c$, hence in this limit the SV degenerates into the usual collapse-unstable Townes soliton. According to the estimate of actual physical parameters presented after Eqs. (2),(3), where $N = 1$ corresponds to $\sim 10^3$ atoms, $N_{\text{max}} \approx 5.85$ implies that the number of atoms in the soliton is limited by $\sim 10^4$.

In the opposite limit of $N \to 0$, the nonlinear terms in Eqs. (2) and (3) become vanishingly small, and the ground state degenerates into a quasi-plane-wave with vanishing amplitudes, radial wavenumber $1/L_{\text{soc}}$, and chemical potential $\mu_0 = -\lambda^2/2$ (cf. Eqs. (4) and (16)). In accordance with this expectation, Fig. 1(c) shows that $N_+/N \to 1/2$ at $N \to 0$. In this limit, the strongly-localized Gaussian ansatz (18) becomes inaccurate, and a full numerical solution is required.

### 3.2. MIXED MODES

Another type of 2D localized vortical states supported by the system of Eqs. (2) and (3) can be initiated by the following input for the imaginary-time simulations, which may also serve as the variational ansatz:

\[
\begin{align*}
(\phi_+^0)_{\text{MM}} &= B_1 e^{-\beta_1 r^2} - B_2 r e^{-i\theta - \beta_2 r^2}, \\
(\phi_-^0)_{\text{MM}} &= B_1 e^{-\beta_1 r^2} + B_2 r e^{i\theta - \beta_2 r^2}.
\end{align*}
\]

Spinors generated by this input are called mixed modes (MMs), as they are built as superpositions of SV-like states with topological charges $(0, -1)$ and $(0, +1)$ in the two components. Although, unlike the SVs, it is not possible to find an exact representation for these modes similar to that given by Eqs. (12) and (13), the numerical and variational results clearly demonstrate their existence. Moreover, as we will see later in this paper, they play the role of the ground state of the system at sufficiently strong cross-attraction between the spin components, $\gamma \geq 1$. In accordance with the form of ansatz (20), the MM is transformed into itself by mirror reflection (14).
Figure 2(a) demonstrates profiles of the MM vortices.

As expected, $\mu(N)$ dependence for the MM states in Fig. 2(b) corroborates the Vakhitov-Kolokolov criterion $d\mu/dN < 0$ too, and the MMs do not require any finite $N$ for their existence either. Thus, the MM states exist in the interval of $N < N'_c = 2N_c/(1 + \gamma)$, where $N_c$ is the above-mentioned critical norm corresponding to the Townes solitons. Indeed, in the limit of $N \to N'_c$ the vortical components vanish in the MM, and it degenerates into the two-component Townes soliton, similar to the above-mentioned degeneration of the SV. At $N \to 0$, the effect of nonlinearities almost vanishes, and the MM state degenerates into a quasi-plane-wave with $\mu \to -\lambda^2/2$. Evidently, this degeneracy is common for both SV and MM modes.

The insertion of input (20), as the variational ansatz, into the functional (7) yields

$$E_{\text{MM}} = \pi \left( \frac{B_1^2}{B_2} + \frac{B_2^2}{\beta_2} - (1 + \gamma) \left( \frac{B_1^4}{4\beta_1^2} + \frac{B_2^4}{32\beta_2^2} \right) - \frac{B_1^2B_2^2}{(\beta_1 + \beta_2)^2} + \frac{8\lambda B_1B_2\beta_1}{(\beta_1 + \beta_2)^2} \right),$$

with norm $N = \pi \left( \frac{B_1^2}{\beta_1} + \frac{B_2^2}{2\beta_2^2} \right)$. The conditional energy minimization with $\partial (E_{\text{MM}} - \mu N)/\partial (B_{1,2}, \beta_{1,2}) = 0$ and fixed $N$ predicts the soliton parameters. Within the variational approach, the boundary in the $(\lambda, \gamma, N)$—parametric space between the MM and SV can be predicted by Eqs. (19) and (21) when the corresponding minimal energies taken for given $(\lambda, \gamma, N)$—set become equal such that:

$$\min \{ E_{\text{MM}}(\lambda, \gamma, N) \} = \min \{ E_{\text{SV}}(\lambda, \gamma, N) \}.$$  \hspace{1cm} (22)

As seen in Fig. 2(a), peak positions of components $|\phi_+(x, y)|$ and $|\phi_-(x, y)|$ in the MM are separated along $x$, Fig. 2(d) showing the separation $\Delta X$ as a function of the norm. For a small amplitude of the vortex component $A_2$, Eq. (20) yields $\Delta X \approx A_2/\alpha_1 A_1$. The separation vanishes as $N$ approaches the aforementioned critical value $N'_c$, at which the MM degenerates into the two-component Townes solitons. This is explained by the fact that, as said above, the vortical components of the wave functions, which cause the shift of the peaks from the center, vanish in this limit.

Figures 1(a) and 2(a) demonstrate the effective radius $\alpha_{1,2}^{-1/2} \sim \beta_{1,2}^{-1/2} \sim 4$ for both solitons, as expected from their common fundamental physics. The scaling of units presented after Eqs. (2),(3), implies the physical size $\sim 3 \mu m$ [66, 67], being of the order of the spin precession length $L_{\text{soc}}$.

### 3.3. EXCITED STATES

In addition to the two types of the ground states, SVs and MMs, numerical analysis reveals their excited varieties. First, a set of excited states can be constructed
as a generalization of the SV pattern given above by Eq. (12):

\[
\phi_+ (x, y, t) = e^{-i\mu t + im\theta_0} r^m f_1(r^2), \quad \phi_- (x, y, t) = e^{-i\mu t + i(m+1)\theta_0} r^{m+1} f_2(r^2),
\]

with integer \( m \geq 1 \). The substitution of this ansatz into Eqs. (2), (3) leads to a system of equations for \( f_1, f_2 \):

\[
\mu f_1 + 2r^2 f_1'' + 2(1 + m) f_1' + r^2 \gamma f_2^2 f_1 - 2\lambda [r^2 f_2' + (1 + m) f_2] = 0,
\]

\[
\mu f_2 + 2r^2 f_2'' + 2(2 + m) f_2' + r^2 \gamma f_1^2 f_2 + 2\lambda f_1' = 0.
\]

In the case of \( m = 0 \), Eq. (24) is tantamount to Eq. (13) for the SV. Mirror-image partners of these states (23) can be generated by transformation (14).

The excited state with \( m = 1 \) in Eq. (23) was found by the imaginary-time integration with the following input:

\[
\phi_0^+ = A_1 r e^{i\theta} e^{-\alpha_1 r^2}, \quad \phi_0^- = A_2 r^2 e^{2i\theta} e^{-\alpha_2 r^2}.
\]

Figure 3(a) shows cross-section profiles for an example of this excited state, obtained with \( N = 5 \) and \( \gamma = 0 \). In Fig. 3(c), the branch of these excited-state solutions is labeled “1”, as it contains vorticity \( m_+ = 1 \) in component \( \phi_+ \).

Another type of excited states was generated by the input with combined vorticities, \( m_+ = 1, -2 \) and \( m_- = -1, 2 \), therefore it is labeled “12” in Figs 3(c):

\[
\phi_0^+ = A_1 r e^{i\theta} e^{-\alpha_1 r^2} - A_2 r^2 e^{-2i\theta} e^{-\alpha_2 r^2}, \quad \phi_0^- = A_1 r e^{-i\theta} e^{-\alpha_1 r^2} + A_2 r^2 e^{2i\theta} e^{-\alpha_2 r^2}.
\]
This state is built by a vortex with topological charge 1 centered at the origin and surrounded by three $-1$ vortices. The contour plot of $|\phi_+ (x, y)|$ in Fig. 3(b), symmetric with respect to the $2\pi/3$-rotation, corroborates this interpretation. The pattern resembles a lattice-like state for the spin-orbit-coupled BEC with the self-repulsive interactions, trapped in the harmonic oscillator potential [68].

3.4. THE GROUND STATE AND STABILITY OF THE VORTICAL MODES

The four types of the vorticity-carrying self-trapped modes, generated by inputs (12), (20), (25), and (26), respectively, can be produced by the imaginary-time integration of Eqs. (2), (3) for any value of the cross-interaction $\gamma$ [in addition to the two latter modes, excited states of still higher orders can be found too – e.g., those given by Eq. (23) with vorticity $m > 1$ – but they all are strongly unstable]. To identify the system's ground state, the total energies of the four species of the vortical modes, calculated as per Eq. (7), and denoted as $E_0$ (for the semi-vortices), $E_{01}$ (for the mixed mode), and $E_1$, $E_{12}$ for the excited states (25) and (26), respectively, are displayed vs. $\gamma$ in Fig. 3(c) for the total norm $N = 3.7$. It is found that the energies satisfy relations $E_0 < E_{01} < E_{12} < E_1$ at $\gamma < 1$ and $E_{01} < E_0 < E_1 < E_{12}$ at $\gamma > 1$. Therefore the semi-vortex and mixed state realize the ground state at $\gamma < 1$ and $\gamma > 1$, respectively, while the states labeled “1” and “12” are indeed excited states, separated by a wide energy gap from the competing ground-state modes. The switch of the ground states at $\gamma = 1$ is not surprising, as it corresponds to the Manakov’s nonlinearity [69], with equal inter- and intra-spin species interactions, where the self-attraction is determined by the full density and, therefore, can lead to various degeneracies. The value of $\gamma$ may be readily altered in experiment by means of the Feshbach resonance [30], hence the ground state may be controlled by means of this technique.

The stability of the four species of 2D self-trapped modes constructed above was studied by means of systematic numerical simulations of their perturbed evolution in the framework of Eqs. (2), (3). The results are reported here for the generic case, represented by two values of the cross-attraction coefficient, $\gamma = 0$ and 2, and a fixed norm, $N = 3.7$. The first result is that the SV, which is the ground state at $\gamma = 0$, and the MM, which plays the same role at $\gamma = 2$, are stable against perturbations (not shown here in detail, as their stability manifests itself in an obvious way).

Next, it is interesting to test the stability of the same two species in the cases when they are not ground states, i.e., the SV at $\gamma = 2$, and the MM at $\gamma = 0$. In the former case, we observe in Fig. 4(a) that the SV profile keeps the initial shape from $t = 0$ till $t = 750$, which exceeds 100 diffraction times (that is the times required for the wavepacket diffraction-induced spread of the state) for the present mode. The instability manifests itself in spontaneous motion of the soliton with a nearly constant
velocity, seen in Fig. 4(b), presenting the coordinates \((X(t), Y(t))\) of the peak of \(|\phi_+ (x, y)|\). On top of the mean velocity, the peak position features small-amplitude oscillations, also seen in Fig. 4(b).

The evolution of the mixed mode at \(\gamma = 0\), when it is not the ground state either, is shown in Fig. 5. Panel (a), pertaining to \(t = 50\) and \(t = 500\) (the latter can be estimated to be \(\approx 70\) diffraction times of the present mode), demonstrates that this state is unstable, starting spontaneous motion and losing the original symmetry between \(|\phi_+|\) and \(|\phi_-|\) with respect to transformation (14). By \(t = 500\), the MM rearranges into a state close to a SV. Further, Fig. 5(b) shows the time evolution of amplitudes of the \(\phi_+\) and \(\phi_-\) components (the solid and dashed curves, respectively) for \(0 < t < 750\). Breaking the original symmetry, the amplitude of \(|\phi_+|\) increases toward the value of the amplitude of the \(|\phi_+|\) component of the SV state for the same \(N = 3.7\), while the amplitude of \(|\phi_-|\) falls to become nearly equal to the amplitude of the \(|\phi_-|\) component of the same SV. Figure 5(c) shows a trajectory of the peak position of the \(|\phi_+|\) component. The localized state moves spontaneously, featuring oscillations in the \(x\) direction, while the average velocity in the \(y\)-direction is \(v_y = -0.0175\).

3.5. MOTION AND COLLISIONS OF VORTEX MODES

To understand the steady motion of the vortex modes, we begin with discussion of the respective anomalous velocities. For the SV modes we have by symmetry:

\[
\langle \sigma_x \rangle_{SV} = \frac{2}{N} \iiint \text{Re} (\phi_+^* \phi_-) \, dx \, dy = 0; \quad \langle \sigma_y \rangle_{SV} = \frac{2}{N} \iiint \text{Im} (\phi_+ \phi_-^*) \, dx \, dy = 0
\]

(27)
Fig. 5 – (a) Contour plots of $|\phi_+ (x, y)|$ at $t = 50$ and $t = 500$ for the mixed mode generated by input (20) with $N = 3.7$ at $\gamma = 0$, when this mode is not a ground state. (b) The evolution of amplitudes of the max $|\phi_+ (x, y)|$ and max $|\phi_- (x, y)|$ components (solid and dashed curves, respectively) for $0 < t < 750$. (c) The trajectory of the peak position of $|\phi_+ (x, y)|$ for $0 < t < 500$.

[cf. Eq. (6)], hence the SVs do not feature the anomalous spin-dependent velocity. For the MM, the expectation value $\langle \sigma_x \rangle$ is different from zero, viz.,

$$\langle \sigma_x \rangle_{\text{MM}} = \frac{\pi B^2}{N \beta_1},$$

(28)

in terms of ansatz (20), thus allowing the motion along the $y$–axis for the MMs, as soon as they are prepared. Note that the numerical results displayed in Figs. 4 and 5 clearly corroborate the existence of the propagating modes.

Localized states that move steadily at velocity $v = (v_x, v_y)$ can be looked for in the form of

$$\phi_+ = \phi_+(x - v_x t, y - v_y t, t), \quad \phi_- = \phi_-(x - v_x t, y + v_y t, t).$$

(29)

As the Galilean invariance is broken by the SOC, the steadily propagating solutions cannot by generated by the formal Galilean transformation, in the form of

$$\phi_\pm(r) \equiv \tilde{\phi}_\pm(r) \exp \left( i \frac{v \cdot r}{2} t \right).$$

(30)

The substitution of ansatz (29) into Eqs. (2), (3) leads to the equations written in the moving reference frame for transformed coordinates $x \to x - v_x t$ and $y \to y - v_y t$

in the form:

$$i \frac{\partial \phi_+}{\partial t} - i (v \cdot \nabla) \phi_+ = -\frac{1}{2} \nabla^2 \phi_+ - (|\phi_+|^2 + \gamma |\phi_-|^2) \phi_+ + \lambda \hat{D}_{\downarrow} \phi_-,$$

$$i \frac{\partial \phi_-}{\partial t} - i (v \cdot \nabla) \phi_- = -\frac{1}{2} \nabla^2 \phi_- - (|\phi_-|^2 + \gamma |\phi_+|^2) \phi_- - \lambda \hat{D}_{\uparrow} \phi_+. \quad \text{(31)}$$

In particular, in the case of $v_x = 0$ (where the moving modes exist, as shown below), the formal Galilean transformation (30) casts Eq. (31) into a form that differs from underlying equations (2) and (3) by the presence of terms causing for linear Rabi
mixing of the two components:

\[
\begin{align*}
    i \frac{\partial \phi_+}{\partial t} &= -\frac{1}{2} \nabla^2 \phi_+ - (|\phi_+|^2 + \gamma|\phi_-|^2)\phi_+ + \lambda \tilde{D}^{-1} \phi_+ + \lambda v_y \phi_- , \\
    i \frac{\partial \phi_-}{\partial t} &= -\frac{1}{2} \nabla^2 \phi_- - (|\phi_-|^2 + \gamma|\phi_+|^2)\phi_- - \lambda \tilde{D}^{-1} \phi_+ + \lambda v_y \phi_+ .
\end{align*}
\] (32)

The same Rabi mixing can be imposed in diverse 2D settings [70, 71] by a GHz wave coupling the two underlying atomic states, i.e., the linear mixing by itself represents a physically relevant addition to the basic model. A straightforward impact of the addition of the mixing terms in Eq. (32) is a shift of the edge of the semi-infinite gap (17) in which solitons may exist, from \( \mu = -\lambda^2 / 2 \) to \( \mu < -\left( \lambda^2 / 2 + |\lambda v_y| \right) \).

Stationary solutions to Eqs. (32) can be obtained by means of the imaginary-time evolution method for \( v_y \neq 0 \), but the procedure produces results solely for \( v_x = 0 \). In particular, at \( \gamma = 2 \), when the quiescent MM is the ground state, its moving version, which is displayed in Figs. 6(a,b) for \( N = 3.1 \) and \( v_y = 0.5 \), exists and is stable too. As well as its quiescent counterpart, this mode features the mirror symmetry between the profiles of \( |\phi_+(x,y)| \) and \( |\phi_-(x,y)| \). Figure 6(c) shows the amplitude of the moving MM, \( A = \sqrt{|\phi_+(0,0)|^2 + |\phi_-(0,0)|^2} \), as a function of \( v_y \). The amplitude monotonously decreases with the growth of the velocity, and the mode vanishes at \( v_y = (v_y)_{\text{max}} \approx 1.8 \).

The availability of the stably moving MMs suggests to consider collisions between them and the effects of interference arising due to their superposition. In particular, simulations of Eqs. (2) and (3) were performed for the head-on collision between two solitons, as shown in Figs. 6(a, b), moving at velocities \( v_y = \pm 0.5 \). Figure 7 displays snapshot patterns of \( \sqrt{|\phi_+(x,y)|^2 + |\phi_-(x,y)|^2} \). The collision results in fusion of the two solitons into a single state, of the same MM type, which is spontaneously drifting along direction \( x \) with velocity \( v_x \approx 0.14 \). The drift may be
understood as a manifestation of the anomalous velocity (6) emerging due to spontaneous symmetry breaking in the course of the collision. Strictly speaking, this state is not a steadily moving one, because, as mentioned above, the imaginary-time integration of Eq. (31) does not produce MM solitons with nonzero $v_x$.

Fig. 7 – The evolution of $\sqrt{|\phi_+ (x,y)|^2 + |\phi_- (x,y)|^2}$ in the head-on collision of two mixed-mode solitons with norms $N = 3.1$ moving at velocities $v_y = \pm 0.5$ [the same as those shown in Fig. 6(a,b)]. Contour plots (a), (b), and (c) are presented for time $t$ shown in the Figures.

At $\gamma = 0$, the SV, which, as shown above, is the ground state in the class of quiescent modes in this case, can also move stably, but only in a small interval of velocities $|v_y| \leq (v_y)_{\text{max}}^{(\text{SV})} \approx 0.03$, much smaller than $(v_y)_{\text{max}}^{(\text{MM})}$. Figures 8(a) and (b) show the profiles of $|\phi_+ (x,y)|$ and $|\phi_- (x,y)|$ for the SV with norm $N = 3.7$, moving at velocity $v_y = -0.02$. In addition, Fig. 8(c) displays the evolution of $\sqrt{|\phi_+|^2 + |\phi_-|^2}$ in the cross section of $x = 0$, produced by direct simulations of Eqs. (2), (3) starting from the initial conditions corresponding to Figs. 8(a, b). The localized solution is stably moving at velocity $v_y = -0.02$. In fact, this moving state is similar to the one generated by the spontaneous onset of motion of the unstable quiescent MM at $\gamma = 0$ and the same norm, which rearranges into a state close to the SV, as shown above in Fig. 5.

At $|v_y| > 0.03$, the solution to Eq. (31) produced by means of the imaginary-time-propagation method converges not to a SV, but rather to a MM state, which turns to be stable. Thus, the moving SVs are rather fragile objects, while the MMs are, on the contrary, robust in the state of motion. This difference can be explained in terms of the anomalous velocities given by Eq. (5), which allow the MMs to move with finite velocities parallel to the $y-$axis without distorting their structures, but do not allow the same for the SV states.
4. EFFECT OF COMBINED RASHBA AND DRESSELHAUS SPIN-ORBIT COUPLINGS

Here we review the effect of an additional SOC of the Dresselhaus type on the vortex solitons. Although the anisotropy of the spectrum with respect to the in-plane rotation makes the analysis much more complicated in this case, see Eq. (4), the topological structure of the self-trapped states remains the same, hence we may use the same ansätze as in the absence of the Dresselhaus term to initiate the analysis, and the imaginary-time simulation will be used to produce the full shape of the states [47].

4.1. SEMI-VORTICES AND MIXED-MODE STATES IN THE PRESENCE OF THE DRESSELHAUS SOC

As we have already established, imaginary-time simulations, as well as the analytical variational approximation, produce soliton solutions of the SV and MM types, starting from the Gaussian ansätze written in terms of the polar coordinates, \((r, \theta)\):

\[
(\phi^0_+)_{SV} = A_1 e^{-\alpha_1 r^2}, \quad (\phi^0_-)_{SV} = A_2 e^{i \theta} e^{-\alpha_2 r^2},
\]

\[
(\phi^0_+)_{MM} = B_1 e^{-\beta_1 r^2} - B_2 e^{-i \theta} e^{-\beta_2 r^2}, \quad (\phi^0_-)_{MM} = B_1 e^{-\beta_1 r^2} + B_2 e^{i \theta} e^{-\beta_2 r^2},
\]

as in Eqs. (18) and (20), respectively.

In the presence of the Dresselhaus term, the ground state of the system (2), (3) can be obtained by means of the imaginary-time simulations. To this end, we scale the Rashba coupling to be \(\lambda \equiv 1\) and study the effect of the Dresselhaus parameter \(\lambda_D\). We start with small \(\lambda_D = 0.05\), and, in particular, focus on the distinction between the cases of \(\gamma < 1\) and \(\gamma > 1\), as they produce different ground states (the SV and MM, respectively) in the absence of the Dresselhaus coupling, as shown above.
Fig. 9 – Plots of (a) $|\phi_+ (x,y)|$, (b) $|\phi_- (x,y)|$, (c) phase of $\phi_+ (x,y)$, and (d) phase of $\phi_- (x,y)$ for a semi-vortex soliton at $\lambda_D = 0.05$, $\gamma = 0.9$, and $N = 3.5$. It is worth mentioning that strong deformation of the semi-vortex components along the $x = \pm y$ directions may be anticipated from Eq. (4), which shows a strong change in the spectrum at $k_x = \pm k_y$, induced by the Dresselhaus term in the SOC. Reproduced with permission from [47]. Copyright (2016) by American Physical Society.

Figures 9(a) and (b) display plots of $|\phi_+ (x,y)|$ and $|\phi_- (x,y)|$, and Figs. 9(c) and (d) are plots of the corresponding phases of $\phi_+ (x,y)$ and $\phi_- (x,y)$ for the SV, at $\lambda_D = 0.05$ and $\gamma = 0.9$. Similarly, Fig. 10 shows absolute values and phases of $\phi_+ (x,y)$ and $\phi_- (x,y)$ for a MM state at $\lambda_D = 0.05$ and $\gamma = 1.1$. The well-defined vorticity of $\phi_-$ is seen in Fig. 9 (d), while the phase of the MM structure of $\phi_+$ and $\phi_-$, as shown in Figs. 10(c) and (d), is more complex. In terms of the spectrum (4), the term accounting for the distortion of the density distributions is $4\lambda_D k_x k_y$. The plots of the SV state are symmetric with respect to both diagonals $y = \pm x$, while the components of the MM state are symmetric solely with respect to $y = x$ line.

To address in detail the crucially important effect of the switch between the SV and MM with the increase in the relative cross-attraction strength $\gamma$, Figs. 11(a) and (b) display $|\phi_+ (x,y)|$ and $|\phi_- (x,y)|$ (solid and dashed lines, respectively), as produced by the imaginary-time integration, in diagonal cross sections $y = \pm x$, for $\gamma = 0.90, 0.95, 1.00, 1.05$, and $1.10$. An essential conclusion is that the switch happens, as in the case of $\lambda_D = 0$, exactly at $\gamma = 1$, and this critical value does not depend on $\lambda_D$, as long as the 2D solitons exist, demonstrating the universality of the effect of degeneracy at $\gamma = 1$. 
4.2. THE DELOCALIZATION TRANSITION

The most essential qualitative effect caused by the addition of the Dresselhaus SOC is the disappearance of the localized modes with the increase in $\lambda_D$, which
happens at a critical value, $\lambda_D = \lambda_D^{(cr)}$. The growing Dresselhaus coupling causes the delocalization, rather than the collapse, as the norm of the solitons remains below the above-mentioned threshold necessary for the onset of the 2D wave collapse. To illustrate this effect, Fig. 12(a) displays $|\phi_+(x,y)|$ and $|\phi_-(x,y)|$ in diagonal section $y = -x$ at $\lambda_D = 0.05, 0.10$, and $0.15$ (marked near the plots) for $\gamma = 0$ and $N = 5$. (b) The amplitude of component $|\phi_+(x,y)|$ versus $N$ at $\lambda_D = 0.05, 0.10$, and $0.15$ (marked near the plots) at $\gamma = 0$.

A detailed picture of the delocalization transition is provided by Fig. 12(b), which shows the amplitude (largest value) of $|\phi_+(x,y)|$ as a function of $N$ for the same set of the Dresselhaus parameters. The delocalization is signaled by the drop of the amplitude to very small values at $N < N_{min}(\lambda_D)$ – for instance, with $N_{min}(\lambda_D = 0.05) \approx 3.5$. Thus, the SV solitons exist in the interval

$$N_{min}(\lambda_D) < N < N_{max}(\gamma < 1),$$

where the largest norm,

$$N_{max}(\gamma < 1) \approx 5.85,$$

is the critical value at which the 2D collapse commences, hence no solitons can exist at $N > N_{max}(\gamma < 1)$. In the limit of $N \to N_{max}(\gamma < 1)$, which corresponds to $\mu \to -\infty$, the bimodal SV state degenerates into the fundamental single spin-component Townes soliton, with $N = N_{max}(\gamma < 1)$. Therefore, $N_{max}(\gamma < 1)$ is $\gamma$-independent if $\gamma < 1$, where the ground state is the SV.

The results for the SV and the MM solitons are summarized by Fig. 13, which shows the $(N,\lambda_D)$-regions where the solitons are stable, as verified by direct real-time simulations. In the panel 13(b), the largest norm corresponds to the value (36)
subjected to obvious rescaling,

\[
N_{\text{max}}(\gamma > 1) = \frac{2}{1 + \gamma} N_{\text{max}}(\gamma < 1). \tag{37}
\]

This is explained by the fact that, in the limit of \( N \to N_{\text{max}}(\gamma > 1) \), which again corresponds to \( \mu \to -\infty \), the vortical terms vanish in ansatz (34), and the soliton degenerates, as mentioned above, into a bound state of two identical Townes solitons in both components. Interestingly, the critical Dresselhaus coupling for the semivortices (Fig. 13(a)), is considerably weaker than that for the mixed modes (Fig. 13(b)).

### 5. Zeeman-Split Solitons

Here we address the role of the synthetic Zeeman splitting, which breaks the time-reversal symmetry and populates one spin component while depopulates the other. The ensuing imbalance tends to strongly suppress the effects of SOC and the cross-spin attraction. We will review how this modification impacts both the SV and MM states.

#### 5.1. Analytical Approaches: Strong ZS and Asymptotic Forms of the Wave Function

Stationary solutions of Eqs. (2) and (3) for 2D solitons with real chemical potential \( \mu \) are sought for as \( \phi_{\pm} = \exp(-i\mu t) u_{\pm}(x,y) \), where complex stationary wave functions are determined by equations

\[
\begin{align*}
\mu u_+ &= -\frac{1}{2} \nabla^2 u_+ - (|u_+|^2 + \gamma |u_-|^2)u_+ + \hat{D} u_- - \Omega u_+ , \tag{38} \\
\mu u_- &= -\frac{1}{2} \nabla^2 u_- - (|u_-|^2 + \gamma |u_+|^2)u_- - \hat{D} u_+ + \Omega u_- . \tag{39}
\end{align*}
\]
An analytical approximation can be developed in the limit of large positive $\Omega$, when Eq. (38) demonstrates that the chemical potential is close to $-\Omega$:

$$\mu = -\Omega + \delta\mu, \ |\delta\mu| \ll \Omega. \quad (40)$$

With the spin-down component, $u_-$, vanishingly small in this limit, Eq. (39) simplifies to

$$u_- \approx \frac{1}{2\Omega} \hat{D}^{(+)} u_+, \quad (41)$$

where $\Omega - \mu$ is replaced by $2\Omega$, pursuant to Eq. (40). Then, the substitution of approximation (41) into Eq. (38) leads to the following equation for $u_+$:

$$(\delta\mu) u_+ = -\frac{1}{2} \left( 1 - \frac{1}{\Omega} \right) \nabla^2 u_+ - |u_+|^2 u_+. \quad (42)$$

By itself, Eq. (42) is tantamount to the nonlinear Schrödinger equation, which gives rise to the Townes solitons; then, Eq. (41) generates a small vortex component of the SV complex. A crucially important fact is the necessity to scale out factor $(1 - 1/\Omega)$ in Eq. (42). Due to the smallness of $1/\Omega$, the scaling easily demonstrates that the norm of the SV complex is, in the present case,

$$N = \left( 1 - \frac{1}{\Omega} \right) N_{\text{max}} (\gamma < 1) + O \left( \frac{1}{\Omega^2} \right), \quad (43)$$

where the last term is a second-order correction corresponding to the norm of the small vortex component given by Eq. (41). Thus, Eq. (43) (which is compared to the corresponding numerically found dependence below in Fig. 16) demonstrates that the total norm of the SV soliton, produced by the present approximation, is slightly smaller than the collapse threshold, $N_{\text{max}} (\gamma < 1)$. Thus, the SV soliton still protected against the collapse, is the ground state. The approximation can be readily extended to the more general system with the linear in the momentum spin-orbit coupling [47]. Note that the lowest-order approximation developed here does not give rise to terms including $\gamma$. Therefore, at a sufficiently large $\Omega$, all still stable solitons are the SV modes, where $N_+$ and $N_-$ are not equal. The energy decrease do to the Zeeman term term favors this inequality, irrespective of the value of $\gamma$.

In addition, one can find analytically the asymptotic form of the wave function in the presence of the ZS. For a moderately strong splitting, with $0 < \Omega < 1$, the SV solitons exist at

$$\mu < -\frac{1 + \Omega^2}{2}, \quad (44)$$

as expected, corresponding to the minimum of the spectrum in Eq. (4). Here the asymptotic form of the soliton is more complex than one given by Eq. (16), with the
localization and precession lengths redefined as

\[ R_{SV}^{-1} = \sqrt{-\mu + 1 + \sqrt{\mu^2 - \Omega^2}}, \]  
\[ L_{soc}^{-1} = \frac{\sqrt{-2\mu + 1 + \Omega^2}}{\sqrt{\mu + 1 + \sqrt{\mu^2 - \Omega^2}}}. \]  

(45)  

(46)

Strong ZS, with $\Omega > 1$, replaces existence condition (44) by $\mu < -1$. More specifically, the SV solitons keep the asymptotic form (45) in the semi-infinite interval (44) of the chemical potentials. However, in the additional finite interval appearing in this case,

\[ \frac{-1 + \Omega^2}{2} < \mu < -1, \]  

(47)

the SV soliton exhibits a strong change of its asymptotics: since the ZS suppresses the displacement-dependent spin rotation on the $L_{soc}$ length scale, the radial oscillations vanish at a sufficiently large $L_{soc}$, while the exponential decrease of the solution at $r \to \infty$ takes place at the radial scale

\[ R_{SV} = \frac{1}{\sqrt{-2\left(\mu + 1 + \sqrt{2\mu + 1 + \Omega^2}\right)}}. \]  

(48)

This analytical prediction agrees well with the numerical results, as shown in Fig. 14(b).

Fig. 14 – (a) Profiles of $|\phi_+(x,0)|$ (solid lines) and $|\phi_-(x,0)|$ (dashed lines) at $\Omega = 1.1, 1.8$, and 2.1 for $\gamma = 0, N = 3$. (b) Profiles of $|\phi_+(r)|$ and $|\phi_-(r)|/r$ at $\Omega = 1.8$, shown on the log scale. The dashed straight line corresponds to the asymptotic exponential form, with the decrease scale predicted by Eq. (48). Reproduced with permission from [47]. Copyright (2016) by American Physical Society.
5.2. SEMI-VORTEX STATES

We begin with the SVs, which, as shown below, are more immune to the action of the Zeeman splitting than the MM states, by applying a variational approximation, similar to Ref. [46]. Using for this purpose the Gaussian ansatz (33) and substituting it into expression (7) yields the total energy as a function of parameters of the ansatz, $A_{1,2}$ and $\alpha_{1,2}$:

$$E_{SV} = \pi \left[ \frac{A_1^2}{2} - \frac{A_1^4}{8\alpha_1} + \frac{A_2^2}{2\alpha_2} - \frac{A_2^4}{64\alpha_2^3} - \frac{\gamma A_1^2 A_2^2}{4(\alpha_1 + \alpha_2)^2} \right]$$

$$+ \frac{4A_1 A_2 \alpha_1}{(\alpha_1 + \alpha_2)^2} + \Omega \left( \frac{A_2^2}{4\alpha_2^2} - \frac{A_2^4}{4\alpha_2^2} \right),$$

(49)
different from Eq. (19) only by setting $\lambda = 1$ and including the Zeeman term. Then, the variational approximation predicts values of the four parameters for the SV soliton as a point at which the energy (49) attains a minimum, subject to the constraint of keeping the total norm fixed.

A family of the SV solitons was produced, in parallel, by means of the imaginary-time simulations of Eqs. (2), (3) and via the variational approach. Figure 14(a) shows the profiles of $|\phi_+ (r)|$ and $|\phi_- (r)|$ at $\Omega = 1.1, 1.8$, and $2.1$ for $\gamma = 0, N = 3$. Figure 14(b) compares the profile of $|\phi_+ (r)|$ and the asymptotic exponential form with the decrease scale predicted by Eq. (48). One can see that with the increase in $\Omega$, the soliton spreads out. Eventually, the amplitude vanishes at some critical ZS strength, $\Omega = \Omega_{cr}$, with only delocalized states existing at $\Omega > \Omega_{cr}$. Figures 15(a) and (b) display this trend by showing the amplitudes of (a) the larger (spin-up) component, $|\phi_+ (r)|$, and (b) the smaller (spin-down) component, $|\phi_- (r)|$, as a function of $\Omega$ for $\gamma = 0$ and $N = 3$. At these parameters the delocalization sets in at

$$\Omega_{cr} (N = 3) \approx 1.95,$$

(50)
while the variational approximation predicts $\Omega_{cr}^{(var)} (N = 3) \approx 1.83$.

Figure 16 presents a summary of the numerical results, showing the stability region for SVs in the $(N, \Omega)$-domain for $\gamma = 0$. Although the plot is limited to $N \leq 5.25$, the analytical result given above by Eqs. (41)-(43) suggests that the SV existence boundary in Fig. 16 extends to $\Omega \to \infty$ in the limit of $N \to N_{max} (\gamma < 1) \approx 5.85$.

5.3. MIXED-MODE STATES AND THE MIXED-MODE - SEMI-VORTEX TRANSITION

As well as in the case of the SVs, the increase of ZS strength $\Omega$ leads to the reduction of the amplitude of the spin-down component $\phi_-$ of the MM soliton, in comparison with its spin-up counterpart, $\phi_+$, as shown in Fig. 17. However, it is
Fig. 15 – (a) The amplitude of the larger (spin-up) component, $|\phi_+ (r)|$, in the SV state as a function of strength $\Omega$ of the Zeeman splitting, for $\gamma = 0$ and $N = 3$. (b) The same for the smaller (spin-down) component, $|\phi_- (r)|$. In both panels (a) and (b), chains of rhombuses and dashed lines show, severally, numerical results and their counterparts produced by the variational approximation based on the minimization of energy (49) at a given norm $N$.

Fig. 16 – The critical strength of the Zeeman splitting, $\Omega_{cr}$, up to which the semi-vortex solitons persist, versus their norm $N$, for $\gamma = 0$ (produced by direct numerical calculation).

Also observed in Fig. 17 that, instead of the delocalization, the MM undergoes a transformation into a stable SV soliton – even at $\gamma > 1$, when solely the MM states, but not SVs, may be stable in the absence of the ZS. Thus, it is relevant to identify the shift of the MM-SV conversion from point $\gamma = 1$, which was the universal boundary between the SV and MM types of the ground state at $\Omega = 0$, to $\gamma > 1$. This shift can be predicted by means of the variational approximation, using ansatz (34) and $\lambda = 1$. The substitution of the ansatz into Eq. (7) yields exactly the same expression as Eq. (21). The energy produced by ansatz (34), does not contain $\Omega$ (in contrast with its SV counterpart (49)) because Eq. (34) implies equal norms of $\phi_+$ and $\phi_-$, hence the
Fig. 17 – Profiles of the spin-up and spin-down components, $|\phi_+(x,0)|$ and $|\phi_-(x,0)|$ (shown by continuous and dashed lines, respectively), of mixed-mode solitons for the strength of the Zeeman splitting $\Omega = 0.05, 0.10, 0.15,$ and $0.20,$ at $\gamma = 1.5$ and $N = 3$. Eventually, the mixed mode transforms into a semi-vortex. Reproduced with permission from [47]. Copyright (2016) by American Physical Society.

Fig. 18 – Stability diagram. The chain of rhombuses shows the numerically obtained value of the cross-attraction strength, $\gamma$, at which the ground state switches from the mixed mode to the semi-vortex, as a function of the Zeeman splitting $\Omega$, for the norm $N = 3$. The dashed line is the same dependence, as predicted by the variational approximation, see Eq. (52). The vertical line at $\Omega \approx 1.95$ is the Zeeman splitting at which the semi-vortex suffers the delocalization, see Eq. (50). Reproduced with permission from [47]. Copyright (2016) by American Physical Society.

respective spin component, which determines the Zeeman energy, vanishes:

$$\langle \sigma_z \rangle_{MM} = \frac{1}{N} \iint (|\phi_+|^2 - |\phi_-|^2) \, dx \, dy = 0. \quad (51)$$
As a result, the difference \( N_+ - N_- \) in the SV state may lead to the realization of the ground state even at \( \gamma > 1 \). Based on Eq. (22), here in the form
\[
\min \{ E_{\text{MM}}(\Omega, \gamma, N) \} = \min \{ E_{\text{SV}}(\Omega, \gamma, N) \},
\]
the variational approximation accurately predicts the value of \( \gamma(\Omega) \), corresponding to the \( \text{MM} \rightarrow \text{SV} \) transition seen in Fig. 18.

The vertical dashed line in Fig. 18, which bounds the stability area of the SVs, corresponds to the critical value of \( \Omega \), given by Eq. (50), at which the SV with norm \( N = 3 \) suffers the delocalization. This value obtained above for \( \gamma = 0 \), actually pertains to all \( \gamma \)'s. Indeed, close to the delocalization transition, the amplitudes of the spin components, \( \max |\phi_-| \) and \( \max |\phi_+| \) are strongly different with \( \max |\phi_-| \ll \max |\phi_+| \), as shown in Fig. 15. Hence, at the delocalization point, the cross-interaction of opposite spin components is much weaker than the self-interaction contribution, having no effect on the transition.

6. CONCLUSION

We have presented a review of recently reported results that show how to construct several types of self-trapped vortex-soliton complexes in the 2D model of the binary BEC, with the spin-orbit coupling (SOC) between the two components and attractive intrinsic nonlinearity. The most essential feature is that, on the contrary to the commonly known instability of 2D free-space solitons and vortices in previously studied models with the attractive cubic terms, two species of stable modes are supported by the spin-orbit coupling. These are semi-vortices (SVs) and mixed modes (MMs), which represent the system’s ground state when the self-attraction is, respectively, stronger or weaker than its cross-interaction counterpart. Experimentally, the SVs and MMs can originally be produced in a trapping potential. Then, the trap is adiabatically lifted, carrying the initial states into the stable free-space self-trapped ones. Under typical experimental conditions, these solitons can hold up to \( \sim 10^4 \) atoms, while their characteristic size is expected to be \( \sim 3 \mu \text{m} \) and determined by the characteristic spin precession length due to the SOC.

The general structure of the ground and excited SV-like states has the exact form presented in Eqs. (12), (13) and (23), (24). The SVs and MMs turn into the known unstable Townes solitons when norms of these states approach the corresponding limit, while there is no lower threshold value of the norm necessary for their existence. While the Galilean invariance is lifted due to the anomalous spin-dependent velocity proportional to the SOC strength, moving stable modes exist up to the respective critical velocities. Collisions between moving mixed modes lead to their fusion.

Next, the generic realization of the combined Rashba-Dresselhaus SOC was
reviewed, and the joint effect of the Zeeman splitting (ZS) and the Rashba term was analyzed too. In this case, families of SV and MM solitons have also been constructed both numerically and analytically. The increase in the strength of the Dresselhaus coupling preserves the soliton type (SV or MM) and eventually leads to its delocalization. The sufficiently strong ZS converts the MM solitons into the SV ones with unequal population of the spin components, which also eventually suffer delocalization. Existence regions have been reviewed for both soliton species. These results help to predict novel experimentally relevant possibilities for the creation of stable vorticity-bearing solitons in matter-wave settings, offered by the generic synthetic SOC and ZS. These results are useful for the understanding of 2D solitons in other nonlinear systems, e.g., in optical waveguides [57, 58].

A challenging possibility for the extension of the reviewed analysis is to study the effects of the Zeeman splitting on metastable 3D SOC-supported solitons found in Ref. [59]. It may also be relevant to apply it to extensively studied optical lattices. Another interesting generalization with an access to new dynamical regimes is the “nonlinearity management” [72] by a time-dependent Feshbach resonance switching the system between $\gamma < 1$ and $\gamma > 1$, where the two different types of the ground state are expected, viz., the SVs and MM$s$, respectively. Another process of the intermode switching can be driven by a time-dependent synthetic Zeeman field.

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