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MUD THROWN FROM A WHEEL

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Abstract. The mud thrown from a rolling wheel problem is a classical, yet very instructive one. By revisiting this problem we find a pedagogical way to connect basic concepts of kinematics in a more quantitative correlation, suitable for teaching in the mechanics class. Two equivalent approaches, one based on a relation between the velocities of two points fixed on a rigid body, the other strengthening the versatility of cycloidal motion, are introduced and compared.

Key words: Kinematics, Classical Mechanics.

1. INTRODUCTION

Learning by solving problems that address common concepts of different subjects or topics in physics helps the student to ordering information and create new connections for the knowledge process. The discrete and continuum approach of elastic properties of materials [1–3], the discrete and continuum linear momentum approach for pulsejet systems [4–6] or for convective velocity in the atmospheric layers [7], the deterministic *versus* random behavior of the gravitational pendulum [8] or coin tossing [9, 10], the quantum theory in connection with the dynamics of photosynthetic system [11–13], the charge *versus* spin currents of electrons [14–17], the quantized *versus* the classical picture of crystal lattice oscillations [18, 19] are just a few examples of a large category in which there are such connections between topics, useful in both research and physics education.

Usually, physics study begins with mechanics, which offers the basic simplest models to describe and predict the behavior of a physical system. As such, kinematics is the first studied topic of physics and this certainly is of great importance in the formation of scientific knowledge skills. This fact motivates the present pedagogical 'crossover correlation' between the kinematics of a rigid body (RB) and the simpler kinematics of a point mass (PM). Our goal is to offer a new simplified (regarding the complexity of notions) view on the motion in the problem by seeing the rolling of the wheel as a planar RB motion. The classical textbook problem we consider next is about the mud thrown from the periphery of a rolling over ground wheel. The usual solution uses geometric intuitive relations for the motion [20–22]. Firstly, we present a solution based on the kinematic equations of a RB, which introduces a more general treatment than that intuitive one used in [20–22]. Secondly, by identifying the motion on the rim of a wheel as a cycloidal motion, we provide the solution based on the PM kinematics on a cycloid.

The problem we are going to discuss in this work is as follows: 'A wheel of radius R is rolling over level ground at a constant speed v_C, without slipping. A piece of mud (PM) breaks loose from the rim of the wheel. What is the greatest height h_{max}, above the ground that a piece of mud can reach?'

The paper is structured as follows. In Sec. 2 the kinematics parameters (position and velocity) of a geometrical point located on the rim of a rolling wheel seen as a RB and the cycloidal trajectory curve described by the same geometrical point are presented. Then, the ballistic motion (without drag) of PM after its detachment from the wheel is discussed starting with the position and velocity obtained from the two approaches. In Sec. 3 we conclude our work.

2. KINEMATICS OF THE MOTION

The simplified model suggested by the text of the problem stated in Introduction is taken into account to describe the motion of the PM detaching from the rolling wheel. The strategy is to compare the PM speed field for two scenarios, one instant, when PM belongs to the rolling wheel seen as a RB, and the other one, when PM is tracked in its motion on the cycloid described by the rim (of the wheel).

2.1. ROLLING WHEEL AS A RIGID BODY

In the kinematics of RB, the velocity of two points *P* and *I* of the body *at a moment*, relative to a fixed system of reference (SR), is given by Euler's formula [23]

$$\mathbf{v_P} = \mathbf{v_I} + \boldsymbol{\omega} \times \mathbf{r_{IP}}, \tag{1}$$

where $\mathbf{v_P}, \mathbf{v_I}, \boldsymbol{\omega}, \mathbf{r_{IP}}$ are the velocities, angular velocity of the rotating SR rigidly attached to the RB and the position vector of *P* relative to *I*, respectively. For given velocities $\mathbf{v_P}$ and $\boldsymbol{\omega}$, Euler's formula states the existence of a point that does not necessarily belong to the RB that has zero velocity. This point is called the instantaneous center of rotation (ICR).

In our problem the wheel is rolling without slipping on the horizontal ground and, in this case, the speed of the center *C* of the wheel, v_C, according to the notations from Fig. 1, is related to the angular velocity of the wheel by the relation

$$v_C = |\mathbf{v_C}| = R\dot{\theta} = -R\omega. \tag{2}$$

Indeed, the relation between the unit vectors $\mathbf{i}'$ $\mathbf{j}'$ of $x'y'$ and $\mathbf{i}$, $\mathbf{j}$ of xy, by using the rotation matrix (or instructively, geometrical considerations), is (see Fig. 1):

$$\begin{pmatrix} \mathbf{i}' \\ \mathbf{j}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix}, \tag{3}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{i}'$, $\mathbf{j}'$ are the unit vectors of xy and $x'y'$, respectively. In Fig. 1, the position of P is defined by the angle between CP and y axis, the angle θ (or equivalently by the angle between CP and x axis, $\theta+\frac{\pi}{2}$ in this case) measured in the counterclockwise direction. The time derivative of (3) for $\mathbf{i}'$ or $\mathbf{j}'$ gives the desired result. For example, (a) $\dot{\mathbf{i}}' = -\dot{\theta}(\sin\theta\mathbf{i}+\cos\theta\mathbf{j}) = -\dot{\theta}\mathbf{j}'$. From the definition of the angular velocity $\boldsymbol{\omega}$ as the time derivative of a vector of RB having constant magnitude, we can write $\dot{\mathbf{i}}' = \boldsymbol{\omega}\times\mathbf{i}'$ and consequently (b) $\dot{\mathbf{i}}' = \omega\mathbf{k}'\times\mathbf{j}'$. By comparing (a) and (b) we obtain indeed $\omega = -\dot{\theta}$. As θ increases in time, ω is negative and (2) is valid.

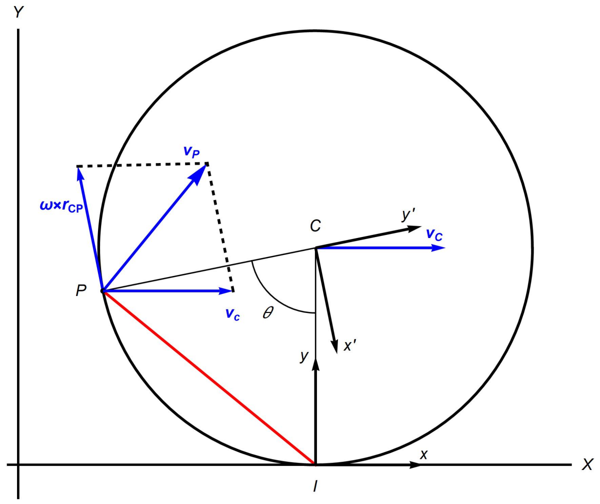


Fig. 1 – (Color online) Rolling without sliding wheel as a planar RB.

By taking I identical with ICR, *instantaneously* I becomes the origin of a fixed xy SR, and we can write the components of the position vector $\mathbf{r_{IP}}$ and velocity $\mathbf{v_P}$ of P on the rim of the wheel relative to this fixed SR as follows:

$$\begin{aligned} x_p &= -R\sin\theta, \\ y_p &= R(1-\cos\theta), \end{aligned} \tag{4}$$

and from $\mathbf{v_P} = \boldsymbol{\omega}\times\mathbf{r_{IP}} = \omega\mathbf{k}\times(x_P\mathbf{i}+y_P\mathbf{j}) = \omega(x_P\mathbf{j}-y_P\mathbf{i})$,

$$\begin{aligned} v_{Px} &= v_C(1-\cos\theta), \\ v_{Py} &= v_C\sin\theta, \end{aligned} \tag{5}$$

with v_{Px} always positive.

As shown in Fig. 1, by adapting (1) we also can write $\mathbf{v_C} = \boldsymbol{\omega} \times \mathbf{r_{IC}}$ and $\mathbf{v_P} = \mathbf{v_C} + \boldsymbol{\omega} \times \mathbf{r_{CP}}$, hence $\mathbf{v_P} = \boldsymbol{\omega} \times (\mathbf{r_{IC}} + \mathbf{r_{CP}}) = \boldsymbol{\omega} \times \mathbf{r_{IP}}$, which is the same above expression of $\mathbf{v_P}$. It is also instructive to mention that the time derivative of Cartesian coordinates from Eqs. (4) does not generate the velocity components from Eqs. (5). We should keep in mind that *xy* is an instantaneous fixed SR and that *I*, in our problem (without slipping), moves relatively to the ground with the velocity of center *C*. Regarding the ballistic motion, PM detaches from the wheel in *P* and the parabola relative to *xy* written as a function of the detaching angle θ and the coordinates x_P, y_P of *P* is

$$y(x,\theta) = y_P\theta + (x - x_P\theta)\frac{v_{Py}\theta}{v_{Px}\theta} - \frac{g(x - x_P\theta)^2}{2v_{Px}^2\theta}. \tag{6}$$

2.2. CYCLOIDAL MOTION

A change of coordinates from *xy* to *XY* by the relations

$$\begin{aligned} x &= X - R\theta, \\ y &= Y, \end{aligned} \tag{7}$$

also transforms Eq. (4) to

$$\begin{aligned} x_P &= X_P - R\theta, \\ y_P &= Y_P, \end{aligned} \tag{8}$$

and we can write

$$\begin{aligned} X_P &= R(\theta - \sin\theta), \\ Y_P &= R(1 - \cos\theta). \end{aligned} \tag{9}$$

According to Eqs. (9), *P* describes a cycloidal in *XY* (see Fig. 2). Next, instead of translating with respect to ICR (the point *I*) the origin of *X* by $-R\theta$ (for each value of θ) we fix the origin of *XY* with respect to the ground and use Eqs. (9) for the coordinates of *P*.

Moreover, the time derivative relative to *XY* (fixed SR) of Eqs. (9) results in the velocity components

$$\begin{aligned} V_{PX} &= \dot{X}_P = R\dot{\theta}(1 - \cos\theta), \\ V_{PY} &= \dot{Y}_P = R\dot{\theta}\sin\theta, \end{aligned}$$

and by taking $R\dot{\theta} = v_C$ (see Eq. (2)) we obtain PM has the same expression of velocity in *xy* and *XY*

$$\begin{aligned} V_{PX} &= v_{Px} = v_C(1 - \cos\theta), \\ V_{PY} &= v_{Py} = v_C\sin\theta. \end{aligned} \tag{10}$$

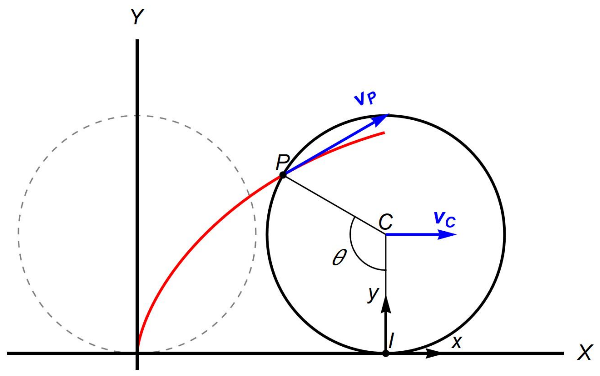


Fig. 2 – (Color online) Cycloidal motion of *P*.

In other words, in *xy* SR the velocities of any *P* at a moment are given by Eq. (5) by choosing its corresponding angle θ, while in *XY* the velocities of *P* are obtained by following the angle θ on the cycloid that starts at $\theta = 0$ and is described by the wheel having the velocity $\mathbf{v_C}$ of the center. Thus the comparison of the two velocity fields mentioned at the beginning of Sec. 2 (the second phrase) is done.

The next task is to obtain the ballistic trajectory of the PM in *XY*. This is easily done either with Eq. (6) by using the transformation from Eq. (7) or by writing the ballistic parabola with the initial conditions from Eqs. (9). Thus one obtains

$$Y(X,\theta) = Y_P(\theta) + (X - X_P(\theta))\frac{v_{PY}(\theta)}{v_{PX}(\theta)} - \frac{g(X - X_P(\theta))^2}{2v_{PX}^2(\theta)}. \quad (11)$$

2.3. BALLISTIC MOTION OF PM

The question of the problem can easily be answered by either using the velocity or trajectory expressions we obtained (Eqs. (6), (10) or (11)). With less (however elementary) calculus the height *P* reaches can be calculated from the kinematics of motion in the vertical direction as

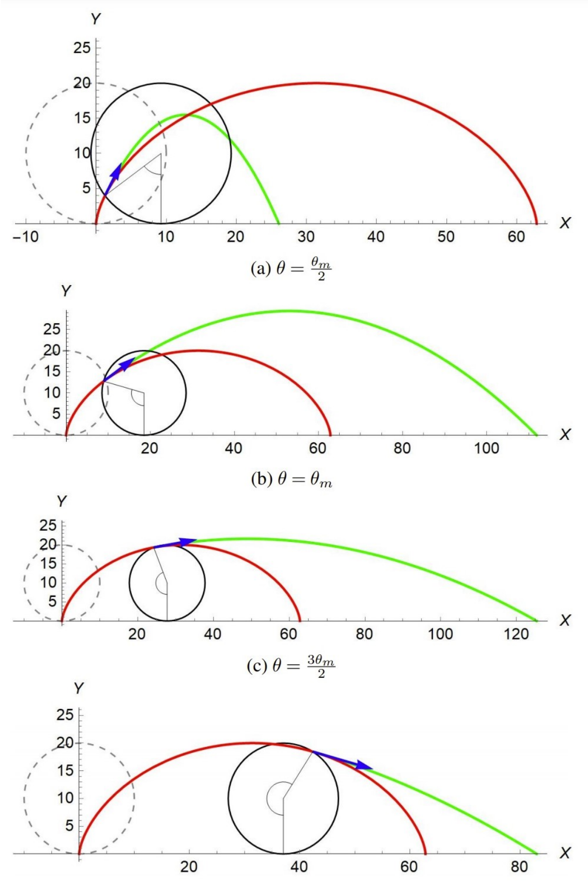
$$y_m(\theta) = \frac{v_{Py}^2(\theta)}{2g} + y_P(\theta), \quad (12)$$

where we considered the notations from the motion in *xy* SR. The maximum value of $y_m(\theta)$ that we denote by h_{max} (as mentioned) is obtained for a certain angle that we denote by θ_m. This angle θ_m is obtained with the extremum of $y_m\theta$ from Eq. (12). Thus from the solution of $\frac{d(y_m(\theta))}{d\theta} = 0$ emerges as $cos\theta_m = -\frac{Rg}{v_C^2}$ and

$$h_{max} = y_m(\theta_m) = R + \frac{R^2 g}{2v_C^2} + \frac{v_C^2}{2g}, \quad (13)$$

which recovers the result from Refs. [20–22].

Next, to provide a more intuitive image of the motion, we present some relevant graphs. In Fig. 3 several parabolas of PM are obtained for different detaching angles θ and the same velocity of the center of the wheel. Figure 3(b) shows the maximum height parabola obtained for the detaching angle θ_m.



(a) $\theta = \frac{\theta_m}{2}$

(b) $\theta = \theta_m$

(c) $\theta = \frac{3\theta_m}{2}$

(d) $\theta = 2\theta_m$

Fig. 3 – (Color online) The ballistic trajectory (green color) and the cylocid (red color) for the motion of PM detaching from the wheel at different angles. The parameters of motion are as follows: $v_0 = 6$ $\frac{\mathrm{m}}{\mathrm{s}}$, $g = 1$ $\frac{\mathrm{m}}{\mathrm{s}^2}$, $R = 10$ m; for these values $\theta_m = 1.852$ rad.

In Fig. 4 the motion of PM and the wheel positions are represented. The black and red dots showing the positions of PM and the center of the wheel, respectively, have a left to right dot to dot correspondence along the X axis.

Using Mathematica software, we also created an animation of the PM motion on the cycloid, on the rim of the wheel, and along the ballistic parabola; the param-

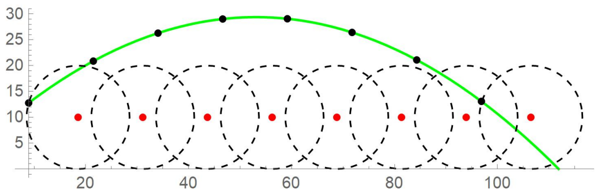


Fig. 4 – (Color online) The ballistic trajectory (green color and black points) and the wheel center (red points) after detaching of P. Detaching of P takes place at the black dot with the smallest value of x coordinate (the leftmost black point), which corresponds to θ_m. The parameters of motion are the same as in Fig. 3.

eters can be changed by the user when accessing https://github.com/SG-722/Mud-thrown-from-a-wheel.git.

3. CONCLUSIONS

In the present work, we described two different methods, one based on the kinematic of the RB , the other on the kinematics of one particle approach, besides to the textbook one, to solve a simple, instructive problem. Thus we could correlate the field velocities of a moving point in two inertial SRs.

With the aim of giving a better understanding of the physical phenomena, we created an animation that illustrates the evolution in time of the PM, before and after it detaches from the moving wheel.

REFERENCES

1. H. Davies, J. Appl. Phys. **84**, 1358 (1998).
2. T.O. Cheche and Y.-C. Chang, J. Appl. Phys. **104**, 083524 (2008).
3. T.O. Cheche and Y.-C. Chang, J. Phys. Chem. A **122**(51), 9910–9921 (2018).
4. W. Johnson, P.D. Soden, and E.R. Trueman, J. Exp. Biol. **56**, 155 (1972).
5. T.O. Cheche, Eur. J. Phys. **38(3)**, 025001 (2017).
6. M. Dolineanu and T.O. Cheche, Rom. Rep. Phys. **71**, 903 (2019).
7. C.V. Vraciu, Rom. Rep. Phys. **73**, 704 (2021).
8. S. Micluta-Campeanu and T.O. Cheche, Nonlinear Dyn. **89**, 81 (2017).
9. R.C. Stefan and T.O. Cheche, Rom. Rep. Phys. **69**, 904 (2017).
10. A. Craciun and T.O. Cheche, Rom. Rep. Phys. **72**, 106 (2020).
11. V. Nagarajan, W.W. Parson, D. Davies, and C.C. Schenk, Biochemie **32**, 12324 (1993).
12. T.O. Cheche and S.H. Lin, Phys. Rev. E **64**, 061103 (2001).
13. T.O. Cheche, M. Hayashi, and S.H. Lin, J. Chin. Chem. Soc. **47**, 729–739 (2013).
14. S. Murakami, N. Nagaosa, and S. C. Zhang, Science **301**, 1348 (2003).
15. T.O. Cheche, Phys. Rev. B **73**, 113301 (2006).
16. T.O. Cheche and E. Barna, Appl. Phys. Lett. **89**, 042116 (2006).

17. D. Ionescu, M. Kovaci, Rom. Rep. Phys. **69**, 501 (2017).
18. S.M. Girvin and K. Yang, *Modern Condensed Matter Physics*, Cambridge University Press (2019), pp. 78–96.
19. T.O. Cheche, EPL (Europhysics Letters) **86**(6), 67011 (2009).
20. G.R. Fowles and G.L. Cassiday, *Analytical Mechanics*, 7th ed., Thomson Brooks/Cole, Belmont, 2005, p. 180.
21. N.D. Newby, Am. J. Phys. **45**, 1116 (1997).
22. F.O. Goodman, Am. J. Phys. **63**, 82 (1995).
23. P.P. Teodorescu, *Mechanical Systems, Classical Models*, Springer (2007), pp. 321–330.