CHAOTIC AND SOLITONIC SOLUTIONS FOR A NEW TIME-FRACTIONAL TWO-MODE KORTEWEG-DE VRIES EQUATION

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Abstract. The two-mode Korteweg-de Vries (TMKdV) equation is a nonlinear dispersive wave model that describes the motion of two different directional wave modes with the same dispersion relations but with various phase velocities, nonlinearity, and dispersion parameters. In this work, we study the dynamics of the model analytically in a time-fractional sense to ensure the stability of the extracted waves of the TMKdV equation. We use the fractional power series technique to conduct our analysis. We show that there is a homotopy mapping of the solution as the Caputo time-fractional derivative order varies over $(0, 1]$ and that both waves have the same physical shapes but with reflexive relation.

Key words: Caputo time-fractional derivative, Two-mode Korteweg-de Vries equation, Fractional power series.

1. INTRODUCTION

There are many recent research studies regarding the mathematical physics dynamical models involving fractional-order derivatives instead of integer-order derivatives. For example, the dynamics of dissipative solitons in the framework of a one-dimensional complex Ginzburg-Landau equation of a fractional order has been explored in [1]. Also, in a recent work [2], in the framework of the nonlinear fractional Schrödinger equation, the asymmetric, symmetric, and antisymmetric soliton solutions have been found and their stability features have been studied numerically. The parity-time symmetric optical modes and the phenomenon of spontaneous symmetry breaking in the space-fractional Schrödinger equation have been investigated by ade-
quate numerical methods [3]. The dynamics of a nonlinear fractional model responsible for the transition of turbulence phenomena and cellular instabilities to chaos has been studied in [4]. For other interesting recent works in the area of fractional models in many physical contexts, see, for example, Refs. [5–9].

Four decades ago, Hirota and Satsuma [10] investigated the interaction of two long waves having different dispersion parameters. They reported that if there is no effect of one wave on the other one, then these two waves will coincide and act as a single wave with a single dispersive parameter; on other words, these waves obey the Korteweg–de Vries (KdV) equation. As a result, the two-mode KdV equation was first established in [11] to reflect the dynamics of moving two different wave modes propagating in the same direction. Later in [12], the two-mode KdV equation has been reformulated in a scaled form to have the following form

\[
\left( D_t^2 - s^2 D_x^2 \right) w + \left( D_t - \alpha s D_x \right) w w_x + \left( D_t - \beta s D_x \right) w_{xxx} = 0, \tag{1}
\]

where \( D_t = \frac{\partial}{\partial t}, D_x = \frac{\partial^2}{\partial x^2}, w = w(x,t) \) is a field function, \( s \) is the phase velocity, \( \alpha \) is the nonlinearity parameter, and \( \beta \) is the dispersive parameter with \( s \geq 0, |\alpha| \leq 1, |\beta| \leq 1 \). In [13], by means of the simplified Hirota’s method, tanh/coth method, and the tan/cot method, different set of solutions to (1) with distinct physical structures are obtained. The conservation laws are used in [14] and a quasi-soliton behavior is reported to the two-mode KdV equation. In [15] and [16], the \((G'/G)\)-expansion method and the Jacobi elliptic function method are implemented to extract more new solitary wave solutions of Eq. (1).

The aim of the current work is to revisit the two-mode KdV equation (1) and to investigate the effect of replacing the integer-order derivative of the time coordinate with fractional-order derivative. The new model under investigation is of the following form

\[
\left( D_t^{2\sigma} - s^2 D_x^2 \right) w + \left( D_t^\sigma - \alpha s D_x \right) w w_x + \left( D_t^\sigma - \beta s D_x \right) w_{xxx} = 0, \quad \sigma < 1, \tag{2}
\]

where now \( D_t^\sigma \) is defined as

\[
D_t^\sigma w(x,t) = \frac{\partial^\sigma w(x,t)}{\partial t^\sigma} = \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-\tau)^{-\sigma} \frac{\partial w(x,\tau)}{\partial \tau} d\tau. \tag{3}
\]

In this context, the model given in Eq. (2) is proposed for the first time, to the best of our knowledge. The fractional power series technique [17–23] will be used to extract the analytical supportive approximate solutions. Next, we will explain the necessary steps of applying the fractional power series to solve Eq. (2) subject to the initial conditions

\[
w(x,0) = f(x), \quad D_t^\sigma w(x,0) = g_i(x): \quad i = 1, 2, \tag{4}
\]
where \( g_1(x) \) is the initial velocity for the first-mode wave and \( g_2(x) \) is the initial velocity for the second-mode wave.

We should point out here that the type of the fractional derivative considered in this work is of Caputo sense. However, many new definitions have been proposed as modifications of Caputo fractional derivative and have been implemented in many works. For example, the Caputo-Fabrizio derivative has been applied to the groundwater flow within Confined Aquifer [24]. Also, the Liouville-Caputo (LC), Caputo-Fabrizio (CF), and Atangana-Baleanu (AB) fractional-order time derivatives have been considered to study the time-fractional versions of both KdV and Burgers’ equations [25, 26]. Fractional derivatives with non-local and non-singular kernels have been established and implemented in many applications [27]. Finally, the new properties of the new conformable derivative have been studied in Ref. [28].

The original contribution of the current work is twofold. We investigate the physical structures of the two-mode KdV equation by means of the elegant Kudryashov method. Then, we study the stability behavior under time evolution of the derived two waves when considering the Caputo time-fractional derivative in the model under investigation.

The organization of this paper is as follows. In Sec. 2, we illustrate how to utilize the fractional power scheme for producing supportive approximate solution of the time-fractional two-mode KdV equation. The solitonic behavior of the two-waves solution of the TMKdV equation is discussed in Sec. 3. Then, in Sec. 4, we validate the proposed scheme by testing a few numerical examples. Finally, we summarize in Sec. 5 the main results obtained in this work.

2. THE FRACTIONAL POWER SERIES METHOD

First, we write the solution of (2) in the following form

\[
    w(x, t) = \sum_{j=0}^{m} \lambda_j(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} + R_m(x, t),
\]

(5)

where \( R_m(x, t) = \sum_{j=m+1}^{\infty} \lambda_j(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} \). We require that \( R_m(x, t) \) is to be very small for \( n \geq m + 1 \) over the intervals \((x, t) \in (a, b) \times (0, T) : \quad T < 1 \). Accordingly, we rewrite (5) as

\[
    w(x, t) = \sum_{j=0}^{m} \lambda_j(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)}.
\]

(6)
By Caputo’s derivative, the fractional derivative of order $0 < \sigma < 1$ for the exponent function has the following rule
\[
D_t^\sigma t^\beta = \begin{cases} 
\frac{\Gamma(\beta + 1)}{\Gamma(\beta - \sigma + 1)} \Gamma(\beta - \sigma + 1) t^{\beta - \sigma}, & \beta \geq \sigma \\
0, & \beta = 0.
\end{cases}
\]
(7)

By (7) and (6), it is easy to conclude the following
\[
D_t^{2\sigma} w(x,t) = \sum_{j=0}^{m-2} \lambda_{j+2}(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)},
\]
(8)

Next, we implement the relations given in (6)-(8) to be inserted in (2) to reach at
\[
L(x,t,\sigma, m) = \sum_{j=0}^{m-2} \lambda_{j+2}(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} - s^2 \sum_{j=0}^{m} \lambda_j^\prime(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)}
\]
\[
+ (D_t^\sigma - \alpha s D_x) \left( \sum_{j=0}^{m} \lambda_j^\prime(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} \right) \left( \sum_{j=0}^{m} \lambda_j^\prime(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} \right)
\]
\[
+ (D_t^\sigma - \beta s D_x) \sum_{j=0}^{m} \lambda_j^\prime(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} = 0,
\]
(9)

where
\[
ww_x = \left( \sum_{j=0}^{m} \lambda_j(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} \right) \left( \sum_{j=0}^{m} \lambda_j^\prime(x) \frac{t^{j\sigma}}{\Gamma(j\sigma + 1)} \right).
\]

Applying the product of two finite series, we have the following expansion
\[
ww_x = A(x,t) = \sum_{j=0}^{2m} \sum_{i=0}^{j} \left( \frac{\lambda_i(x)\lambda_{j-i}(x)}{\Gamma(i\sigma + 1)\Gamma((j-i)\sigma + 1)} \right) t^{j\sigma}
\]
\[
- \sum_{j=0}^{m-1} \sum_{i=m+1}^{2m-j} \left( \frac{\lambda_i(x)\lambda_j(x)}{\Gamma(i\sigma + 1)\Gamma(j\sigma + 1)} \right) t^{(j+i)\sigma}.
\]
(10)

Therefore, implementing the operator $D_t^\sigma$ on $A(x,t) = ww_x$ leads to
\[
D_t^\sigma (ww_x) = B(x,t) = \sum_{j=0}^{2m-1} \sum_{i=0}^{j+1} \left( \frac{\lambda_i(x)\lambda_{j-i+1}(x)}{\Gamma(i\sigma + 1)\Gamma((j-i)\sigma + 1)} \right) t^{j\sigma}
\]
\[
+ \sum_{j=0}^{m-1} \sum_{i=m}^{2m-j-1} \left( \frac{(\lambda_{i+1}(x)\lambda_j(x))\Gamma((j+i+1)\sigma + 1)}{\Gamma((i+1)\sigma + 1)\Gamma(j\sigma + 1)\Gamma((j+i)\sigma + 1)} \right) t^{(j+i)\sigma},
\]
(11)
Now, we combine (8) with the resulting formulas in (9), (10), and (11) to reach at the following function

\[
L(x,t,\sigma,m) = \sum_{j=0}^{m-2} \lambda_{j+2}(x) \frac{t^{j+1}}{(j+1)!} - s^2 \sum_{j=0}^{m} \lambda_j^\sigma(x) \frac{t^j}{(j+1)} + B(x,t) + \alpha s \frac{\partial A(x,t)}{\partial x} + \sum_{j=0}^{m-1} \lambda_{j+1}^\sigma(x) \frac{t^j}{(j+1)} - \beta s \sum_{j=0}^{m} \lambda_j^\sigma(x) \frac{t^j}{(j+1)} = 0.
\] (12)

It is clear from (6) that \( \lambda_0(x) = w(x,0) \) and \( \lambda_1(x) = D_t^2 w(x,0) \). Thus, to determine \( \lambda_m(x) : m = 2, 3, 4, \ldots \), we solve, recursively, the following equation

\[
D_t^{(m-2)} L(x,t,\sigma,m) = 0, \quad m = 2, 3, 4, \ldots.
\] (13)

3. SOLITONIC SOLUTIONS FOR THE TWO-MODE KDV EQUATION

We aim in this Section to seek some solutions for the two-mode KdV equation (1) to show the physical structures of such type of nonlinear equations. Commonly in similar studies, we seek the solution of the form

\[
w(x,t) = \sum_{k=0}^{n} a_k Y^k, \quad Y = \tanh(\mu(x-ct)) \text{ or } \coth(\mu(x-ct)).
\] (14)

The above suggested solution is derived from the well-known tanh-coth-function method [29–34]. Substitution of (14) in (1) and collecting the coefficients of \( Y^i \) yields a system of nonlinear algebraic equations in terms of the unknowns \( a_k, \mu, \) and \( c \). But the resulting system can not be solved unless we add a reasonable extra conditions. We may assume that \( \alpha = \beta = h \). Therefore, six different solutions are obtained:

\[
w(x,t) = a_0 + a_1 \tanh(\mu(x-ct)), \quad h = \pm 1,
\] (15)

\[
w(x,t) = a_0 + a_1 \coth(\mu(x-ct)), \quad h = \pm 1,
\] (16)

and

\[
w(x,t) = 4\mu^2 - 12\mu^2 \tanh \left( \mu(x + (2\mu^2 \pm \sqrt{s^2 - 4hs\mu^2 + 4\mu^2}t)) \right),
\] (17)

\[
w(x,t) = 4\mu^2 - 12\mu^2 \coth \left( \mu(x + (2\mu^2 \pm \sqrt{s^2 - 4hs\mu^2 + 4\mu^2}t)) \right), \quad |h| \leq 1.
\] (18)
and

\[ w(x, t) = 12\mu^2 - 12\mu^2 \tanh \left( \mu(x - (2\mu^2 \mp \sqrt{s^2 + 4hs\mu^2 + 4\mu^4})t) \right), \quad (19) \]

\[ w(x, t) = 12\mu^2 - 12\mu^2 \coth \left( \mu(x - (2\mu^2 \mp \sqrt{s^2 + 4hs\mu^2 + 4\mu^4})t) \right), \quad |h| \leq 1. \quad (20) \]

The above parameters \( a_0, a_1, \) and \( \mu \) are free parameters. Figure 1 represents the solitonic behavior of the two-waves depicted in (15) and their phase interactions upon increasing the phase velocity \( s \). In the next Section, we study the propagation of these moving two-waves if the integer-order time derivative is replaced by the fractional-order time derivative.

![Fig. 1 – The interaction of the two waves depicted in (15) upon increasing the phase velocity \( s \), where \( a_0 = a_1 = \mu = 1 \) and \( s = 1, 3, 5 \), respectively.](image)
4. DYNAMICS OF THE TIME-FRACTIONAL TWO-MODE KDV EQUATION

In this Section, we apply the fractional power series scheme, which was introduced earlier, to solve the following time-fractional version of two-mode KdV equation

\[
(D_t^{2\sigma} - s^2 D_x^2) w + (D_t^{\alpha} - \alpha s D_x) w w_x + (D_t^{\beta} - \beta s D_x) w_{xxx} = 0, \quad \sigma < 1, \tag{21}
\]

subject to

\[
w(x, 0) = f(x), \quad D_t^\sigma w(x, 0) = g_i(x) : \ i = 1, 2, \tag{22}
\]

where \(g_1(x)\) and \(g_2(x)\) are regarded as the initial velocities, for the right-mode wave and left-mode wave, respectively.

In order to investigate the efficiency of the proposed method and then to explore the physical structure of the new model given in (21), we study the following example. Consider the equation (21) subject to

\[
w(x, 0) = \tanh(\mu x), \quad D_t^\sigma w(x, 0) = \ s \ h \ \mu \ \text{sech}^2(\mu x). \tag{23}
\]

Based on the result obtained in (15), the exact solution of the problem (21) and (23) is \(w(x, t) = \tanh(\mu(x - sht))\) when \(\alpha = \beta = h: \ h = \pm 1\) and the fractional order is \(\sigma = 1\). Now, following the structure of the fractional power series illustrated in (12) and (13), we present the first four terms of the sequence \(\{\lambda_i(x)\}_{i=0}^\infty\)

\[
\begin{align*}
\lambda_0(x) &= \tanh(\mu x), \\
\lambda_1(x) &= s \ h \ \mu \ \text{sech}^2(\mu x), \\
\lambda_2(x) &= -2s^2 \mu^2 \ \text{sech}^2(\mu x) \ \tanh(\mu x), \\
\lambda_3(x) &= \frac{\Gamma^2(1+\sigma)(\sinh(3\mu x) - 3 \sinh(\mu x)) + 2 \Gamma(1+2\sigma) \sinh(\mu x)}{\Gamma^2(1+\sigma)/(s^2 \mu^4 \ \text{sech}^4(\mu x))}.
\end{align*}
\tag{24}
\]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\(x/t\) & 0.10 & 0.15 & 0.20 & 0.25 & 0.30 & 0.35 \\
\hline
-0.5 & 2.77 \times 10^{-10} & 4.63 \times 10^{-8} & 3.39 \times 10^{-7} & 1.57 \times 10^{-6} & 5.46 \times 10^{-6} & 1.55 \times 10^{-5} \\
-0.3 & 5.81 \times 10^{-10} & 1.25 \times 10^{-8} & 1.13 \times 10^{-7} & 6.28 \times 10^{-7} & 2.58 \times 10^{-6} & 8.51 \times 10^{-6} \\
-0.1 & 4.80 \times 10^{-9} & 8.32 \times 10^{-8} & 6.31 \times 10^{-7} & 3.04 \times 10^{-6} & 1.09 \times 10^{-5} & 3.24 \times 10^{-5} \\
0.1 & 4.44 \times 10^{-9} & 7.4 \times 10^{-8} & 5.39 \times 10^{-7} & 2.5 \times 10^{-6} & 8.67 \times 10^{-6} & 2.26 \times 10^{-5} \\
0.3 & 2.81 \times 10^{-12} & 2.27 \times 10^{-9} & 3.38 \times 10^{-8} & 2.38 \times 10^{-7} & 1.11 \times 10^{-6} & 4.02 \times 10^{-6} \\
\hline
\end{tabular}
\caption{Table 1}
\end{table}

To this end, we consider \(w_0(x, t) = \sum_{i=0}^{6} \lambda_i(x) \ \frac{\Gamma^i(\sigma)}{(\sigma+1)^i}\) being the supportive
Fig. 2 – Contour plots of $w_6(x,t)$ when $h = \pm 1$ and $a_0 = a_1 = \mu = s = \sigma = 1$.

Fig. 3 – (c) The right-mode wave "$h = 1$" $w_6(x,t)$. (d) The left-mode wave "$h = -1$" $w_6(x,t)$. (e) Both right-left waves are depicted in $w(x,t)$. 

Original two-mode waves, $\mu = 1$, $s = 1$, $\sigma = 1$
approximate solution of (21) and (23). Now, we are ready to give some graphical analysis regarding the obtained approximate solution. Figure 2 represents the contour plots of the two waves of $w_6(x,t)$ for the case of $\sigma = 1$. We observe that both waves have the same physical shapes and the reflexive relation. For clarity, we give the name of right-mode wave when $h = 1$ and left-mode wave when $h = -1$.

To study the efficiency of the proposed method we present the plots of both left and right-mode waves depicted in $w_6(x,t)$ against the exact solution $w(x,t)$, see Fig. 3. Moreover, the obtained absolute errors evaluated at some mesh points are given in Table 1. This comparison is achieved by comparing "in particular" the right-mode wave $w_6(x,t)$ against the exact solution $w(x,t)$ when $h = 1$.

![Fig. 4 – (I) Profile solutions of $w_6(x,t)$ when $h = 1$. (II) Profile solutions of $w_6(x,t)$ when $h = -1$.](image)

Finally, Fig. 4 shows the effect of the fractional order $\sigma$ on the propagation of the two-mode waves. It can be seen that there is a homotopy mapping of the solution as $\sigma$ varies over $(0,1]$. For large values of $\sigma$, this homotopy preserves the same physical shape. In other words, the solution is stable when $\sigma$ approaches one and coincides with the integer-case derivative when $\sigma = 1$.

5. CONCLUSION

In this work we have introduced a new physical model, namely the two-mode KdV equation that describes the propagation of two directional symmetrical waves moving simultaneously. We have generalized this dynamical model to include the fractional-order time derivative instead of the common integer-order time derivative. The fractional power series method was used to find a supportive approximate solution for this fractional physical model.

The validity and accuracy of the proposed method was verified using illustrative graphs and a supporting Table. We have concluded this study by examining the effect of the fractional-order derivative on ensuring the stability of the corresponding
waves of the two-mode KdV equation. As a future work, we aim to apply the methodology of this work to study other types of fractional two-mode evolution equations.

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