DYNAMICS OF INTEGER-FRACTIONAL TIME-DERIVATIVE FOR THE NEW TWO-MODE KURAMOTO-SIVASHINSKY MODEL

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\textit{Received November 4, 2019}

Abstract. In this paper, we investigate the dynamics of a nonlinear model responsible for the transition of turbulence phenomena and cellular instabilities to a chaos. The two-mode Kuramoto-Sivashinsky (TMKS) model is an example of such application. We study both integer and fractional time-derivative involved in this model. Solitary wave solutions and approximate analytical solutions will be derived to TMKS model by means of well-posed different techniques. The mechanism of the concepts of two-mode and time-fractional derivative will be discussed in this work. Finally, both 2-dimensional and 3-dimensional plots will be provided to support our findings.

\textit{Key words}: Two-mode Kuramoto-Sivashinsky (TMKS) model; Kudryashov-expansion method; time-fractional TMKS; Maclaurin series.

1. INTRODUCTION

The two-mode Kuramoto-Sivashinsky (TMKS) model is a physical model used to describe the transition of turbulence phenomena and cellular instabilities to chaos [1, 2]. The time-fractional TMKS equation reads as

\[
(D_t^{2\alpha} - s^2 D_x^2) u + (D_t^{\alpha} - \gamma_1 s D_x) u u_x + (D_t^{\alpha} - \gamma_2 s D_x) (k_1 u_{xx} + k_2 u_{xxxx}) = 0,
\]

where \(D_t^{\alpha}\) is the fractional Caputo derivative of order \(0 < \alpha < 1\), which is defined as

\[
D_t^{\alpha} u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} \, d\tau,
\]

and \(\gamma_1, \gamma_2\) are, respectively, the nonlinearity and dispersion factors with norm less than or equal to one. The factor \(s\) represents the phase velocity of the interaction of two directional waves generated by the above two-mode model [3–8]. The term
$u_{xx}$ stands for the instability at large scales with positive factor $k_1$, whereas the term $u_{xxxx}$ is responsible for damping at small scales with positive factor $k_2$.

In this work, the model given in (1) is proposed for the first time, to the best of our knowledge. Some classical techniques have been modified to handle many fractional models in diverse physical contexts [9–14]. Here, the fractional power series technique [15–21] will be used to extract analytical supportive approximate solutions. We aim to study the role of the time-fractional derivative acting on the dynamics of the propagations of the resulting two-waves of the TMKS model. Next we will explain the steps of using the fractional power series to solve (1) subject to

$$
egin{align*}
  u(x,0) & = f(x), \\
  D_t^\alpha u(x,0) & = g_i(x): \quad i = 1, 2.
\end{align*}
$$

Here $g_1(x)$ is the initial velocity for the first-mode wave and $g_2(x)$ is the initial velocity for the second-mode wave. The organization of this paper is as follows. In Sec. 2, we extract the soliton solutions for the TNKS model and show the dynamics of the obtained solutions. In Sec. 3, we give a brief description of the fractional Maclaurin series scheme. Section 4 is devoted to validating our finding by testing numerical examples of the fractional TNKS model using the proposed technique. Finally, we summarize the benefits gained in this study in Conclusion.

2. SOLUTIONS OF TMKS MODEL BY KUDRYASHOV-EXPANSION TECHNIQUE

We use here the Kudryashov technique to extract possible solutions of the integer-derivative two-mode Kuramoto-Sivashinsky model

$$
  u_{tt} - s^2 u_{xx} + \left( \frac{\partial}{\partial t} - \gamma_1 s \frac{\partial}{\partial x} \right) \{uu_x\} + \left( \frac{\partial}{\partial t} - \gamma_2 s \frac{\partial}{\partial x} \right) \{k_1 u_{xx} + k_2 u_{xxxx}\} = 0.
$$

By the wave variable $z = x - ct$ and the chain rule. Equation (4) is reduced to the following differential equation

$$
  (k^2 - s^2) u - \frac{1}{2} (c + \gamma_1 s) u^2 - (c + \gamma_2 s) \left( k_1 u' + k_2 u'' \right) = 0,
$$

The proposed method requires the solution in the form [22, 23]

$$
  u(x,t) = \sum_{i=0}^{n} \lambda_i Y^i, \quad Y = Y(z), \quad z = x - ct.
$$

The variable $Y$ satisfies the differential equation

$$
  Y'' = \mu Y(Y - 1).
$$
Solving (7) gives
\[ Y(z) = \frac{1}{1 + d e^{\mu z}}, \] (8)
with \( d \) being an arbitrary constant. Now, comparing the order of \( u^2 \) versus \( u''' \) in (5) will lead to the result that \( n = 3 \) and accordingly we write (6) as
\[ u(z) = \lambda_0 + \lambda_1 Y + \lambda_2 Y^2 + \lambda_3 Y^3. \] (9)

Differentiating both (6) and (7) implicitly, leads to
\[ Y'' = \mu^2 Y(Y - 1)(2Y - 1), \]
\[ Y''' = \mu^3 Y(Y - 1)(6Y^2 - 6Y + 1) \] (10)
and
\[ u'(z) = \lambda_1 Y' + 2\lambda_2 YY' + 3\lambda_3 Y^2 Y', \]
\[ u''(z) = \lambda_1 Y'' + 2\lambda_2 YY'' + (Y')^2 + 3\lambda_3(2Y(Y')^2 + Y^2 Y'''), \]
\[ u'''(z) = \lambda_1 Y''' + 2\lambda_2(YY''' + 3Y'Y'') + 3\lambda_3(2(Y')^3 + 6Y Y'' Y''' + Y^2 Y'''). \] (11)

Substitution of (9) through (11) in (5) will convert the problem into a finite series in the variable \( Y \). The coefficients of \( Y^i \) are identical to zeros, and this lead to an algebraic system in the unknowns \( \lambda_2 \), \( \mu \), and \( c \). Solving the resulting system gives the following outcomes
\[ \gamma_1 = \gamma_2, \]
\[ \gamma_2 = 180 k_2 \mu^3, \]
\[ \gamma_2 = -120 k_2 \mu^3, \] (12)
and the following two sets.

**Set I:**
\[ \lambda_0 = 0, \]
\[ \lambda_1 = -\frac{720\sqrt{11/19}k_1^{3/2}}{19\sqrt{k_2}}, \]
\[ \mu = \frac{\sqrt{11/19}k_1}{\sqrt{k_2}}, \]
\[ c = \frac{-15\sqrt{11}k_1^{3/2} + \sqrt{2475k_1^2 + 6859k_2 s^2 - 570\sqrt{209k_1^{3/2}k_2^{1/2} / \gamma_2}}}{19\sqrt{19}\sqrt{k_2}}. \] (13)
Set II:

\[
\begin{align*}
\lambda_0 &= \frac{60\sqrt{11/19k_1^{3/2}}}{19\sqrt{k_2}}, \\
\lambda_1 &= -\frac{720\sqrt{11/19k_1^{3/2}}}{19\sqrt{k_2}}, \\
\mu &= \frac{\sqrt{11/19\sqrt{k_1}}}{\sqrt{k_2}}, \\
c &= \frac{15\sqrt{11k_1^{3/2} + 6859k_2s^2 + 570\sqrt{209k_1^{3/2}k_2^{1/2}k_3^{1/2}}\gamma_2s}}{19\sqrt{119k_2}}. 
\end{align*}
\] (14)

Figure 1 shows the physical structures for the two-waves of the TMKS model depicted in (13) and being classified as right/left-mode waves. Figure 2 shows the dynamics of the propagations of the obtained two waves upon increasing the phase velocity \(s\).

![Figure 1: Right and Left Mode Waves](image)

**3. MACLAURIN FRACTIONAL POWER SERIES METHOD**

To solve the problem given in (1), the fractional power expansion scheme suggests the solution to be of the following form

\[
u(x,t) = \sum_{j=0}^{m} \lambda_j(x) t^{j\alpha} \frac{1}{\Gamma(j\alpha + 1)} + \sum_{j=m+1}^{\infty} \lambda_j(x) t^{j\alpha} \frac{1}{\Gamma(j\alpha + 1)}. \] (15)
Fig. 2 – The interaction of the two waves depicted in (13) upon increasing the phase velocity $s$, where $\gamma_2 = 0.5$, $\mu = d = k_1 = k_2 = 1$ and $s = 0.5, 1, 1.5$, respectively.

We choose $m$ large enough so that the remainder $\sum_{j=m+1}^{\infty} \lambda_j(x) \frac{t^{j\alpha}}{(j\alpha+1)}$ is of reasonable small error in the region $(x,t) \in (a,b) \times (0,T)$: $T < 1$. Therefore, we may rewrite (15) as

$$u(x,t) = \sum_{j=0}^{m} \lambda_j(x) \frac{t^{j\alpha}}{(j\alpha+1)} = \lambda_0(x) + \lambda_1(x) \frac{t^{\alpha}}{(\alpha+1)} + \sum_{j=2}^{m} \lambda_j(x) \frac{t^{j\alpha}}{(j\alpha+1)}, \quad (16)$$

By Caputo-sense and other fractional derivatives types, the rule for differentiating the exponent function is defined as

$$D_t^\alpha t^\beta = \begin{cases} 
\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & 0 < \alpha \leq 1, \beta \geq \alpha \\
0, & 0 < \alpha < 1, \beta < \alpha
\end{cases} \quad (17)$$
Applying (17) on (16), we get

\[ D_t^{2\alpha} w(x,t) = \sum_{j=0}^{m-2} \lambda_{j+2}(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}, \]  

(18)

Next, we substitute the relations (16)-(18) in (1) to get

\[ H(x,t,\alpha,m) = \sum_{j=0}^{m-2} \lambda_{j+2}(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} - s^2 \sum_{j=0}^{m-2} \lambda_j''(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} + \frac{1}{j\alpha + 1}) \left( \sum_{j=0}^{m\alpha} \lambda_j'(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} \right) + (D_t^{\alpha} - \gamma_1 s D_x) \sum_{j=0}^{m\alpha} (k_1 \lambda_j''(x) + k_2 \lambda_j'''(x)) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} = 0, \]  

(19)

where \( uu_x = \left( \sum_{j=0}^{m\alpha} \lambda_j(x) \frac{t^{\alpha}}{\Gamma(j\alpha + 1)} \right) \left( \sum_{j=0}^{m\alpha} \lambda_j'(x) \frac{t^{\alpha}}{\Gamma(j\alpha + 1)} \right) \). Applying the product of two finite series, we have the following expansion

\[ uu_x = A(x,t) = \sum_{j=0}^{m\alpha} \sum_{i=0}^{j} \left( \frac{\lambda_i(x)\lambda_{j-i}(x)}{\Gamma(i\alpha + 1)\Gamma(j-i)\alpha + 1} \right) t^{j\alpha} - \sum_{j=0}^{m-1} \sum_{i=m}^{2m-j} \left( \frac{\lambda_i(x)\lambda_{j-i}(x)}{\Gamma(i\alpha + 1)\Gamma(j-i)\alpha + 1} \right) t^{(j+i)\alpha}. \]  

(20)

Therefore, implementing the operator \( D_t^{\alpha} \) on \( A(x,t) = uu_x \) leads to

\[ D_t^{\alpha}(uu_x) = B(x,t) = \sum_{j=0}^{2m-1} \sum_{i=0}^{j+1} \left( \frac{\lambda_i(x)\lambda_{j-i+1}(x)}{\Gamma(i\alpha + 1)\Gamma((j+i)\alpha + 1)} \right) t^{j\alpha} - \sum_{j=0}^{m-1} \sum_{i=m}^{2m-j-1} \left( \frac{\lambda_{i+1}(x)\lambda_j'(x)}{\Gamma((i+1)\alpha + 1)\Gamma((j+i)\alpha + 1)} \right) t^{(j+i)\alpha}. \]  

(21)
Now, we combine (18) with the resulting formulas in (19), (20), and (21) to reach at the following function

\[
    H(x,t,\alpha,m) = \sum_{j=0}^{m-2} \lambda_{j+2}(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} - s^2 \sum_{j=0}^{m} \lambda''_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} + B(x,t) - \gamma_1 s \frac{\partial A(x,t)}{\partial x} + \sum_{j=0}^{m-1} (k_1 \lambda''_{j+1}(x) + k_2 \lambda'''_{j+1}(x)) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}
    - \gamma_2 s \sum_{j=0}^{m} (k_1 \lambda'''_{j}(x) + k_2 \lambda''''_{j}(x)) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} = 0.
\]

(22)

It is clear from (16) that \( \lambda_0(x) = u(x,0) \) and \( \lambda_1(x) = D^\alpha_t u(x,0) \) and will be provided by the problem under investigation. Thus, to determine \( \lambda_m(x) : m = 2,3,4,... \), we solve the following equation

\[
    D^{(m-2)\alpha}_t H(x,t,\alpha,m) = 0, \quad m = 2,3,4,..
\]

(23)

4. NUMERICAL INVESTIGATIONS FOR THE TIME-FRACTIONAL TMKS EQUATION

Now, we are ready to study the efficiency of the fractional Maclaurin series on solving (1) for some assigned values of the parameters of the TMKS equation. To perform this investigations, we consider the following problem

\[
(D^{2\alpha}_t - D_x^2) u + (D^\alpha_t - D_x) u u_x + (D^\alpha_t - D_x) (u_{xx} + u_{xxxx}) = 0, \quad \alpha < 1.
\]

(24)

subject to

\[
    u(x,0) = \frac{60\sqrt{11/19}(-1 + 9e^{0.760886x} - 12e^{1.52177x})}{19(1 + e^{0.760886x})^3},
\]

\[
    D^\alpha_t u(x,0) = g_i(x): \quad i = 1,2,
\]

(25)

where

\[
    g_1(x) = \frac{21.9391e^{0.760886x} - 76.7867e^{1.52177x} + 21.9391e^{2.28266x}}{(1 + e^{0.760886x})^4},
\]

\[
    g_2(x) = \frac{4.4185e^{0.760886x} - 15.4648e^{1.52177x} + 4.4185e^{2.28266x}}{(1 + e^{0.760886x})^4}.
\]

(26)

For simplicity, we give special names of the obtained two-waves for the fractional TMKS equation, as right-mode wave and left-mode wave, moving with initial velocities of \( g_1(x) \) and \( g_2(x) \), respectively.
In our investigation, the fifth truncated series, $u_5(x,t)$ will be considered the favorite approximate solution of the fractional TMKS equation. Figure 3 shows the right-mode approximate solution, exact solution and their comparison for the case of $\alpha = 1$ subject to the initial velocity $g_1(x)$. Figure 4 shows the left-mode approximate solution, exact solution, and their comparison for the case of $\alpha = 1$ subject to the initial velocity $g_2(x)$. It can be seen that the plotted approximate results are in excellent agreement with the exact solutions.

Regarding the effect of the fractional order in propagating the obtained two waves, profile solutions for different values of $\alpha$ are reported in Fig. 5 for both right-mode and left-mode waves. We may say that the role of the fractional order $\alpha : \alpha \in (0,1]$ plays a homotopy mapping from the initial solution to the exact solution of the problem. Another aspect one can notice that the relations among the depicted profile solutions converge asymptotically to the exact solution as $\alpha$ is increasing to 1. Such an observation is a strong evidence for the stability of our proposed numerical technique.
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5. CONCLUSION

In this paper, an integer-fractional derivative is considered for a new nonlinear model under the name of two-mode Kuramoto-Sivashinsky equation. This model describe the propagations of two directional identical waves moving with different
initial velocities. The interaction of these two waves depends on the values of the model’s phase velocity. Numerical investigations are performed to study both cases of the model, integer-time derivative and fractional-time derivative. Graphical analysis was conducted in this work to show the dynamics of these waves and to study the effect of the fractional order in stabilizing their propagations. As future work, we will extend our approach to investigate more recently developed two-mode equations described in Refs. [24–29].

REFERENCES