Romanian Reports in Physics **71**, 108 (2019)

ON THE PREDICTABILITY OF THE FINAL STATE IN A RING OF KURAMOTO ROTATORS

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Received March 18, 2019

Abstract. Emergence of order in randomly initialized systems of locally coupled identical Kuramoto rotators is studied. Depending on the system's initialization different types of collective oscillation modes (dynamically stationary states) can develop. Following the dynamics of such oscillator ensembles the emergent collective mode can be predicted after a certain time interval. For large enough systems, this prediction time is scaling with the square of the system's size.

Key words: emergent synchronization; Kumamoto model; dynamical systems; basins of attraction; time scaling.

1. INTRODUCTION

The Kuramoto model [1] has become a prototype system for studying collective behavior in complex systems [2]. Starting from the paradigmatic mean field model of Kuramoto many varieties of coupling topologies have been studied up to now: first order neighbors on regular and random graphs [3, 4], neighbors of different order on regular lattices [5, 6], time delayed interactions [7, 8], and oscillators with repulsive coupling [9, 10]. The above-mentioned works have both analytical and numerical results and concentrate mainly on the general properties of the achievable stationary states: phase locking and stability of incoherence. The possibility of reaching chimera states in Kuramoto systems has also been confirmed [11, 12], and given new direction for such studies. Apart of searching for the final collective modes, there are works, which focus on the dynamics of such systems, particularly on how the allowed stationary states are reached [13]. The present paper intends to contribute in this direction by studying the time moment of the dynamics when the final collective state becomes predictable. We consider a system of locally coupled identical Kuramoto oscillators with interaction topology sketched in Fig. 1. The equations of motion for such configuration with N oscillators ($i \in [1, N]$) is given by:

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_0 + K\left[\sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i)\right]. \tag{1}$$

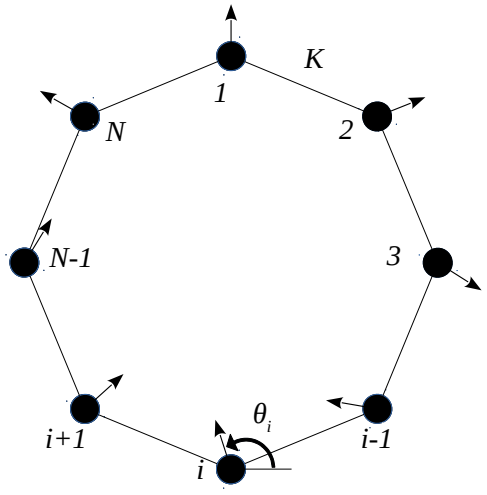

Fig. 1 – (color online) Interaction topology in the considered model. Interactions are between the neighbors connected by the thin lines.

The phase of the oscillators are represented by θ_i , natural frequencies of the rotators are ω_0 and K is the coupling parameter ($K > 0$). In these systems besides the classical synchronous state (no phase-shift between the oscillators) other phase locking states can also appear [6, 14–17]. Due to the extremely complicated basins of attractions of the allowed stationary states [17], it is practically impossible to predict from random initial conditions the stationary state that will be selected by the rotator ensemble. The number of allowed stationary states increases linearly with the systems size and the volume of the state space increases exponentially. As a consequence, predicting the final state from initial conditions becomes an increasingly challenging task for large oscillator ensembles. Recently we proposed a method that allows the prediction of the final stationary state at a given point of the dynamics. Although the method fails in some pathological cases, it allows prediction with an average error less than 0.01%. In the followings we briefly overview the allowed stationary states of the studied system, our prediction method and summarize the results on the estimated average time necessary for such prediction.

2. STABLE COLLECTIVE MODES

For convenience, let us first shift to a reference frame rotating with the angular velocity ω_0:

$$u_i(t) = \theta_i(t) - \omega_0 t. \tag{2}$$

The equation of motions in this reference frame writes as:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = K\left[\sin(u_{i-1}-u_i)+\sin(u_{i+1}-u_i)\right]. \tag{3}$$

In this manner, the system has a form of a gradient system [6] where the equations of motions can be written as the gradient of a potential function defined by the phases of the oscillators. Therefore, the potential will always decrease throughout the dynamics and the the fixpoints of the potential energy surface will correspond to the desired stationary states. It is more convenient now to write up the state of the system in terms of relative positions of the oscillators. For this purpose we introduce a quantity called phase shift defined as follows:

$$\Delta\phi_i = (\phi_i - \phi_{i-1}) - 2\pi \left\lfloor \frac{\phi_i - \phi_{i-1} + \pi}{2\pi} \right\rfloor \tag{4}$$

Here, ϕ_i denotes the phase of the oscillator i mapped on to the $[0, 2\pi)$ interval and $f(x) = \lfloor x \rfloor$ is the floor function. The above defined $\Delta\phi_i$ value behaves now as a signed (oriented) distance on the unit circle between the endpoint of two unit length rotating vectors representing the oscillators. Thus, the phase shift takes values from the $[\pi, \pi)$ interval.

Having a proper measure for the relative phases, the stationary states of Eq. (1) can be easily determined. Using the $\Delta\phi_i$ phase shifts we can identify three types of stationary states (collective oscillation modes) as in [17]. Type **(a)** states represent the states where all phase shifts are equal and constant in time:

$$\Delta\phi_{i+1} = \Delta\phi_i. \tag{5}$$

Periodic boundary conditions imply that:

$$N \cdot \Delta\phi = 2m\pi, \tag{6}$$

where N is the number of oscillators, while $m \in \mathbf{Z}$ is the *winding number*, ranging from $-N/2$ up to $+N/2$ if N is even and $-(N-1)/2$ up to $(N-1)/2$ if N is odd. These states correspond to the minima (stable states) and maxima (unstable states) of the potential energy surface. The winding numbers also encodes the stability of a state. Ordered states with winding number:

$$-N/4 < m < N/4. \tag{7}$$

are stable while all the others are unstable [6, 16–18].

The second branch (type **(b)**) is characterized by the following equation:

$$\Delta\phi_{i+1} = \pm\pi - \Delta\phi_i \tag{8}$$

The sign of π is fixed over the system: it is either positive or negative for all triplets (two phase shifts are made up by three oscillators, hence the triplets). One can notice

that the equation above results in the following alternating pattern over the system: $\Delta\phi,\ \pm\pi - -\Delta\phi,\ \Delta\phi,\ \pm\pi - \Delta\phi,\ \ldots$ Again, due to periodic boundary conditions these states are only possible when N is divisible by 4, and their winding number is $\pm N/4$. Furthermore, one can show that these states are unstable thus their importance is low.

Finally, type **(c)** states are mixtures of the first two cases. Namely, for some triplets of oscillators Eq. (5) is true, while for the remaining triplets Eq. (8) holds. Periodic boundary conditions restrict the number of possible configurations, still it scales exponentially with the number of oscillators (figure not shown). These stationary states are the saddle points of the potential, thus they are unstable, however their role in the dynamics can be important and influences directly the error in our final state prediction method [17].

3. PREDICTION AND SCALING

For a better understanding of the dynamics first let us give a more visual picture of the phase space of this N dimensional system. From the definition of the phase shifts in Eq. (4) one can derive that for each time moment in the dynamics it is true that [17]:

$$\sum_{i=1}^{N} \Delta\phi_i = 2m\pi, \tag{9}$$

where m is the same winding number as in Eq. (6). If we consider the N dimensional space of all phase shifts, which is an N dimensional hypercube, this equation determines parallel hyperplanes. In other words the phase point of the system is always on one of these planes. By this condition the N dimensional space is reduced to a set of $N-1$ dimensional hyperplanes that are confined in a hypercube. Jumps between the planes are also possible since the m winding number also changes during the dynamics. One can consider thus a dynamics in the winding number and a first attempt for predicting the final state would be then to correlate the initial and final winding numbers. In our recent study (see Ref. [17]) we have shown however that the prediction of the final state based on the initial winding number fails since the correlation between the initial and final winding numbers decreases as the system size grows. Therefore predictions based solely on snapshots of the system become more and more unreliable as N increases. To overcome this problem other, more reliable prediction methods have to be developed.

In order to monitor the changes in the state of the system through time, a gen-

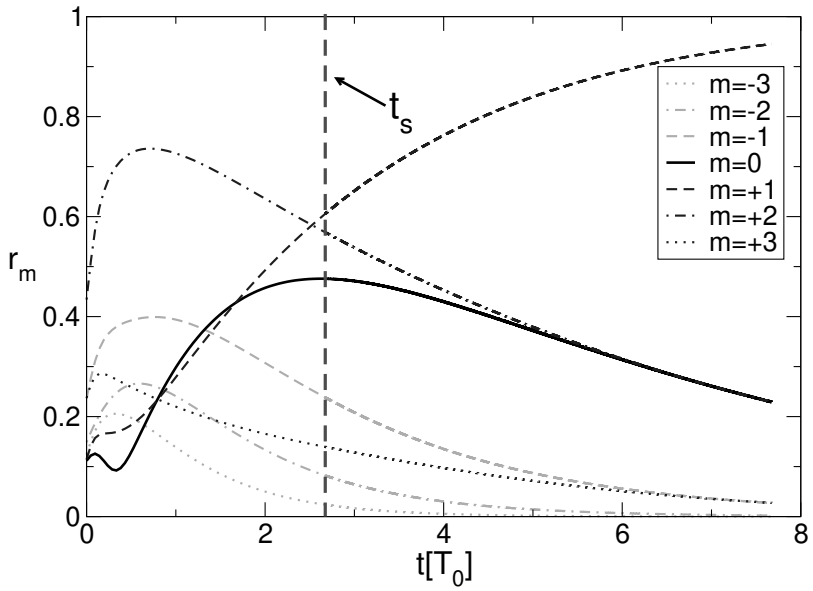

Fig. 2 – (color online) Time evolution of the order parameters for N = 24, K = 1. Order parameters with higher m were suppressed to protect the graph from overcrowding. The vertical dashed line indicates the first time moment when only one order parameter is increasing. This moment marks the beginning of the relaxation process and the state with the increasing parameter will be selected. The selection time t_s is the interval from $t = 0$ up to the moment denoted by the vertical line.

eralized Kuramoto order parameter was introduced [17]:

$$r_m(t)\mathrm{e}^{i\psi_m(t)} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{i[\theta_j(t)-(j-1)\frac{2m\pi}{N}]}. \tag{10}$$

The absolute value of this parameter (r_m) ranges from 0 to 1 and it measures the level of order with respect to type (a) ordered state with a given m winding number. According to the argument presented in Ref. [17] the time series of the order parameters offers a convenient method for predicting the final ordered state at a given moment of the dynamics. To illustrate the argument from Ref. [17], in Fig. 2 we have plotted a particular realization of the dynamics $r_m(t)$, for a system with $N = 24$ rotators. One can distinguish two stages in the dynamics separated by the time moment denoted with t_s. For $t > t_s$ all the generalized order parameters $r_m(t)$ have a monotonic exponential-like relaxation either to 1 or to 0. One of the order parameters is relaxing to 1 (r_1), while all the other generalized order parameters are relaxing to 0. According to our non rigorously proved **hypothesis** we assume in this region of the dynamics no more surprises and the system approaches the already selected stationary state (the one for which the order parameter is increasing). In Ref. [17] it

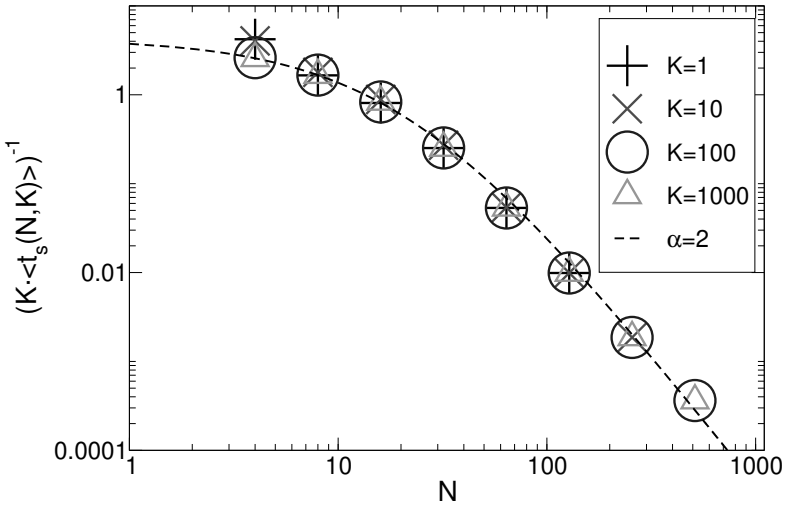

Fig. 3 – (color online) Reciprocal value of the averaged selection time ($K \cdot \langle t_s(N,K) \rangle$) as a function of the system size N . The data for different K values is rescaled. The collapsed data is fitted with a Lomax II distribution type function* with $\alpha = 1$. Due to the large computational time, for weaker couplings ($K = 1$ and $K = 10$) results are available only for smaller N values. Please note the logarithmic scales.

is shown that apart of a few pathological cases indeed this happens. Exceptions are when the system gets very close to a saddle point. It was found that with a probability of 99.99% the final state is predictable when a time moment t_s is identified so that only one order parameter is increasing. On the other hand, for $t < t_s$ the dynamics of the $r_m(t)$ quantities are complicated and usually non-monotonic. The interval from $t = 0$ up to the beginning of the relaxation process is referred as selection time (t_s), implying that the system's final state can be predicted. Please keep in mind here that the actual trajectories are deterministic, meaning that the final states are directly linked to initial positions through the basins of attraction, however the complexity and the dimensionality of these domains makes extremely hard to uncover these connections. This is why such prediction possibility is useful for computational studies when large rotator ensembles are statistically studied. There are many other delicate aspects concerning this final state prediction method left to be discussed. This paper is not intended to enter in deep details concerning these aspects and a more thorough study is performed in Ref. [17].

Here we focus on the scaling of the average selection time as a function of the system size N. Due to the increasing complexity of the basins of attraction for the stationary states, we expect that $t_s \to \infty$ as $N \to \infty$. The actual form of the $t_s(N)$ dependence is what we intend to uncover by computer simulation experiments. From Eq. (2) it is easy to observe that the dependence of t_s as a function of K is trivial,

since changing the K value to K' is equivalent with rescaling the time $t' = t \cdot K/K'$. In our computer experiments we studied a large number of trajectories starting the system from random initial states. We fixed different N and K values and determined the average selection time: $\langle t_s(N,K)\rangle$. Since the coupling rescales the time it is possible to collapse the data for different K and same N value. In order to do this we calculate the average value of $\langle t_s(N,K)\rangle$ for a given K coupling strength taken over different N values: $\langle t_s(K)\rangle = \langle\langle t_s(N,K)\rangle\rangle_N$. Due to the $t \propto K^{-1}$ scaling property we expect that $K \cdot \langle t_s(N,K)\rangle$ will collapse on a universal curve depending solely on N. In Fig. 3 we plot $1/K \cdot \langle t_s(N,K)\rangle$ as a function of the number of oscillators. The collapsed data can be fitted with a Lomax II distribution type function* with the $\alpha = 1$, $\lambda = 21.55$ and $C = 46.24$ parameters. This result suggests that in the limit of large N values the selection time increases as a the square of the system size: $\langle t_s(N,K)\rangle \propto N^2$.

4. CONCLUSIONS

Scaling of the average time needed for predicting the final stationary state of randomly initialized systems of locally coupled identical Kuramoto oscillators was studied. Prediction of the final stationary state was made using a recently introduced method based on the derivatives of a generalized Kuramoto order parameter [17]. We found that the average selection time increases as the square of the number of oscillators. Scaling as function of the coupling strength between the rotators is trivial due to the obvious rescaling property of the time-evolution equations. Our result is in agreement with what one would naturally expect. For infinite system size, there is an infinite time needed in order to observe emergent collective behavior in the system. In such a view, a Mermin-Wagner type theorem [19] corresponding to our one-dimensional dynamical system with local coupling is not violated in the thermodynamic limit. There is an infinite time needed to observe spontaneous symmetry breaking.

Acknowledgements. Work supported by the UEFISCDI grant: PN-III-P4-PCE-2016-0363.

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* Lomax II is a probability density function: $f(x) = C\left[1 + \frac{x}{\lambda}\right]^{-\alpha-1}$, $x \geq 0$, with shape $\alpha > 0$, scale parameter $\lambda > 0$, and normalization constant $C > 0$.

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