Romanian Reports in Physics, Vol. 67, No. 4, P. 1322–1330, 2015

Dedicated to
International Year of Light 2015

SYMMETRY OF PACKING OF DOPED CAVITIES AND ITS INFLUENCE ON THE EMISSION SPECTRUM OF ENTANGLED STATES

NICOLAE ENAKI*, SERGIU BAZGAN

Institute of Applied Physics, Academy of Sciences of Moldova,
Academiei 5, MD-2028, Chisinau, Moldova,
*Corresponding author *E-mail*: enache@asm.md

Received October 1, 2015

Abstract. The packing geometry plays an important role in the molecular systems and its symmetry may be used in many systems of coupled cavities. We are focused on the description of the symmetry of single mode coupled cavities in each of them are placed a two-level atom. It is shown that the number of collective states is reduced with increasing of the local symmetry. In the case, when the system is reduced to the two cooperative distinguished subsystems consisted from atoms and field, the possibility of quantitative description of entanglement between these two subsystems becomes possible. Using the conception of discord and quantum mutual information the entanglement between these subsystems was estimated.

Key words: quantum optics, cavity quantum electrodynamics, symmetry, coupled cavities.

1. INTRODUCTION

Recently, a great attention is devoted to the quantum systems composed from coupled cavities. For example, a model of a chain of atom-microcavity systems coupled by a common fiber optic is proposed in the ref. [1]. The authors have analyzed the possibility to form the super-modes and they study the dependence of this effect on the chain geometry and the number of atoms and cavities. The cooperative states between the radiators placed in two and three cavities are proposed in the Ref. [2]. The cooperative interaction of N two-level radiators with cavity electromagnetic field (EMF) was in the attention of many experimental and theoretical researches [3]. This it is connected with the big application of the two-level system as a q-bit in quantum processing of information. In many cases, distinct ensembles of q-bits are used in the realization of quantum registers [4, 5]. According to the indistinguishability principle for atoms [6], 2^N states of N two-level radiators can be reduced to $N+1$ states in the processes of coherent excitation. As was demonstrated in the

literature the new effects are possible to be observed, when the atoms are placed in the single mode cavity. In this case, the Dicke model opens new peculiarities like the giant quantum oscillators [7], atom-fields entanglement [8] and phase transition [9]. It is not difficult to observe that this number of collective excitations is drastically reduced with the increasing number of atoms. It is attractive from the physical point of view to apply this principle for atoms placed in coupled cavities.

As follows from the distinctive description of doped cavities [10, 11], the analytical description of the quantum systems becomes complicated, due to the increasing with the number of degrees of freedom by increasing with a number of coupled cavities in the system. In this case, the number of degrees of freedom is connected to the number of coupled quantum oscillators: atoms and cavity-modes. Considering that the excitations of atoms and cavity modes become indistinguishable, we can use the symmetry transformation of such a system considering the invariance of the states after the actions of rotations corresponding to the local symmetry group.

Following the conception of Ref. [2], we propose to revise the collective states of atoms and cavity modes in order to reduce them to simple situation of two quantum subsystem in interaction. Here we define the new collective quantum states, which take into consideration the intrinsic symmetry of atomic and field states, according to the packing method used in the construction of coupled cavity systems. In this situation the collective of atoms and cooperative field states in the cavity systems can be reduced to two distinguished quantum sub-system in interaction. Due to the symmetry between two coupled cavity we can introduce the collective modes and atomic states like in the Ref. [2], so that it is possible to calculate the quantum mutual information between the atomic ρ_A and field ρ_F states: $I(\rho) = H(\rho_A) + H(\rho_F) - H(\rho_{AF})$.

2. EXACT SOLUTIONS FOR SCHRÖDINGER EQUATION

The theoretical model described in introduction is developed bellow. Let us start with the Hamiltonian of a system formed of N coupled cavities (see Fig. 1) in which is trapped one two-level atom

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \hat{H}_I, \\
\hat{H}_0 &= \hbar\omega \sum_{i=1}^{n} (\hat{a}_i^+ \hat{a}_i + |e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|), \\
\hat{H}_I &= \hbar g \sum_{i=1}^{n} (|e_i\rangle\langle g_i| \hat{a}_i + h.c.) + \hbar \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \chi_{i,j} (\hat{a}_i^+ \hat{a}_j + \hat{a}_j^+ \hat{a}_i).
\end{aligned} \tag{1}$$

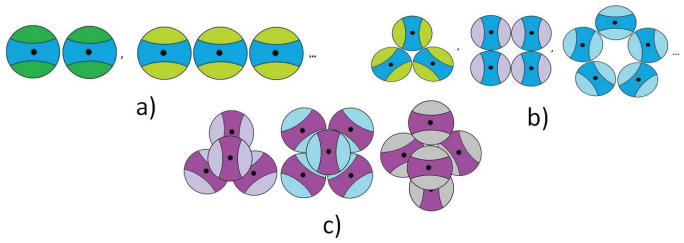

Fig. 1 – Packing of doped coupled cavities a) 1D, b) 2D and c) 3D.

Here ω represents the resonance frequency between one mode of cavity and dipole active transition of atom, $|e\rangle$ and $|g\rangle$ are the atomic excited and ground states, g is the interaction constant between EMF and atom, $\chi_{i,j}$ describes the intensity of coupling between the cavity "i" and "j". We have simplified the problem considering that the interaction constants between atom and EMF are same in all cavities. If the coupling between the cavities is achieved through the evanescent field, then the coupling constants $\chi_{i,j}$ depend on the location of cavities on the system (this coupling exponentially decreases with increasing of the distance between cavities $E = E_0 \exp\{-\alpha x + i\beta y\}$, where α is the attenuation constant, β is the propagation constant). In this situation we can consider that only the neighboring cavities will be coupled. For example, N cavities, we can pack in a line (1-2-3...-N), considering the non zero value only for the coupling constants $\chi_{i,i+1} = \chi$, where $i = 1..N-1$ (the first cavity and last one are uncoupled). If N cavities are packed in the vertices of a regular polygon, then all neighboring cavities are coupled ($\chi_{1,N} = \chi_{i,i+1} = \chi$, $i = 1..N-1$). Starting at $N = 4$ the cavities can be located in space with different symmetry. For example, with total rotation symmetry are 4 cavities situated in the vertices of a regular tetrahedron ($\chi_{i,j} = \chi$, for $i = 1..3$ and $j = (i+1)..4$). If 5 cavities are packed in the vertices of a rectangular pyramid, then the cavity placed in the apex (cavity 5) is coupled to all other cavities (numbered from 1 to 4) and only the neighboring cavities from the base are coupled ($\chi_{1,4} = \chi_{i,i+1} = \chi_1$, $\chi_{j,5} = \chi_2$, for $i = 1..3$ and $j = 1..4$). At the same time, we can pack 5 cavities in the vertices of a triangular bipiramyd (in apex are placed the cavities 4 and 5). In the last situation, the coupling between cavities is led by following symmetric coefficients: $\chi_{1,3} = \chi_{i,i+1} = \chi_1$, $\chi_{j,k} = \chi_2$ for $i = 1,2$; $j = 1..3$ and $k = 4,5$. In special case, we can take $\chi_1 = \chi_2$ for last two presented situations. Method of solving of Schrödinger equation was presented in Ref. [2]. Below we present the possible states for one excitation (atom or field) in the system. As in Ref. [2], we obtain the eigenvalues and eigenstates of the interaction part of Hamiltonian 1.

The eigenvalues and the number of degenerate states of Hamiltonian 1 strongly depend on the symmetry of the system. This symmetry is dictated by the position

of cavities in the case when the coupling between them is realized due to evanescent electromagnetic field. Below we study the influence of symmetry on the solutions of the Schrödinger equation. For simplicity, we start with the description of the system formed of N coupled cavities, placed in vertices of a regular polygon. As was mentioned above, only the neighboring cavities are coupled. If the number of cavities is large, according to the Hamiltonian (1) for one indistinguishable excitation of atoms or photons we observe the following degenerate wave functions

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{N}}\left(|e_1 g_2..g_N\rangle + ... + |g_1 g_2..e_N\rangle\right)|0_1 0_2..0_N\rangle\,, \\ |\psi_2\rangle &= \frac{1}{\sqrt{N}}\,|g_1 g_2..g_N\rangle\left(|1_1 0_2..0_N\rangle + ... + |0_1 0_2..1_N\rangle\right). \end{aligned} \tag{2}$$

If the system is prepared initially in one of these states, during the evolution it remains in the same states superposition. For this, we need to find the superposition coefficients and eigenvalues of the interaction Hamiltonian: $H_I\,|\psi_1\rangle = \hbar g\,|\psi_2\rangle$ and $H_I\,|\psi_2\rangle = \hbar g\,|\psi_1\rangle + 2\hbar\chi\,|\psi_2\rangle$. Taking into consideration the rotation symmetry, we obtain only above two states instead of $2N$ atom - field vector states of a single excitation in N distinguished cavities. The two cooperative eigenvalues of the interaction Hamiltonian are

$$\lambda_{1(2)} = \chi \pm \sqrt{\chi^2 + g^2}. \tag{3}$$

For these two eigenvalues the new wave functions $|\psi\rangle_j = c[j,1]\,|\psi_1\rangle + c[j,2]\,|\psi_2\rangle$ contains the following coefficients:

$$c[j,1] = \sqrt{g^2/(g^2+\lambda_j^2)}, \quad c[j,2] = \lambda_j/\sqrt{(g^2+\lambda_j^2)}, \tag{4}$$

for $j = 1, 2$. The wave function of the system can be presented as a superposition of eigenstates of interaction Hamiltonian

$$|\Psi(t)\rangle = \sum_{j=1}^{n} a_j \exp(i\lambda_j t)\,|\psi\rangle_j \tag{5}$$

If the cavities are packed in line (1-2-3 - ...-N), the number of degenerate states increases substantially. For three cavities, for example, we have 4 states for a single undistinguished excitation: $|\psi_1\rangle = (|egg\rangle + |gge\rangle)\,|000\rangle\,/\sqrt{2}$, $|\psi_2\rangle = |geg\rangle\,|000\rangle$, $|\psi_3\rangle = |ggg\rangle\,(|100\rangle + |001\rangle)/\sqrt{2}$ and $|\psi_4\rangle = |ggg\rangle\,|010\rangle$. Let us for simplicity eliminate the index for cavity number, considering that the position in the state indicates this number. The eigenvalues in this case are

$$\lambda_{1(2)} = \mp\sqrt{\alpha^2 - \chi\varepsilon}, \quad \lambda_{3(4)} = \mp\sqrt{\alpha^2 + \chi\varepsilon}. \tag{6}$$

The following notations are used these eigenvalues: $\varepsilon = \sqrt{2g^2 + \chi^2}$ and $\alpha = \sqrt{g^2 + \chi^2}$. The coefficients for eigenstates are presented in appendix.

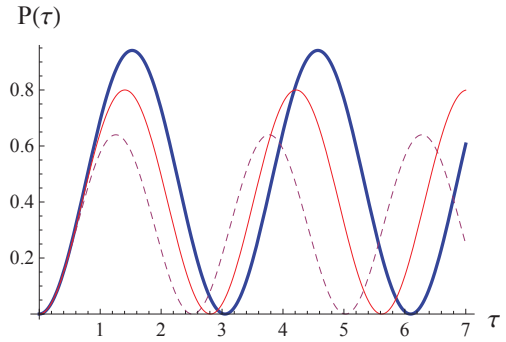

Fig. 2 – Time dependence of photon number for 4 coupled cavities packed in: a) line (thick line), b) vertices of a square (red line) and c) vertices of a tetrahedron (dashed line).

Let us pass to 4 coupled cavities. In this case we have the possibility to pack them in 2D (in line or in the vertices of a regular polygon) or 3D space (in the vertices of a tetrahedron for maxim symmetry). For 4 cavities in line it is obtained: $|\psi_1\rangle = (|eggg\rangle + |ggge\rangle)|0000\rangle/\sqrt{2}$, $|\psi_2\rangle = (|gegg\rangle + |ggeg\rangle)|0000\rangle/\sqrt{2}$, $|\psi_3\rangle = |gggg\rangle(|1000\rangle + |0001\rangle)/\sqrt{2}$ and $|\psi_4\rangle = |gggg\rangle(|0100\rangle + |0010\rangle)/\sqrt{2}$. The eigenvalues for them are $\lambda_{1(2)} = \frac{1}{2}(-\chi \mp \beta)$ and $\lambda_{3(4)} = \frac{1}{2}(\chi \mp \beta)$, where $\beta = \sqrt{4g^2+\chi^2}$. If the cavities are placed at the vertices of a tetrahedron, we have only two states:

$$\begin{aligned} |\psi_1\rangle &= 0.5(|e000\rangle + |0e00\rangle + |00e0\rangle + |000e\rangle)|0000\rangle, \\ |\psi_2\rangle &= 0.5|0000\rangle(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle). \end{aligned} \quad (7)$$

The eigenvalues are $\lambda_{1(2)} = \frac{1}{2}(3\chi \mp \varepsilon)$, where $\varepsilon = \sqrt{4g^2+9\chi^2}$. Using a same method, we have obtained the wave function of the system composed from 5 cavities, placed in line, in vertices of a regular polygon, in vertices of a square pyramid and in the vertices of a triangular bipyramid. We propose to study the influence of the packing symmetry on the time evolution of collective photon number excitations in 4 coupled cavities. In Fig. 2 we plot this dependence considering that: $\chi/g = 0.1$, $|\Psi(0)\rangle = 0.5(|eggg\rangle + ... + |ggge\rangle)|0000\rangle$. The significant influence of the packing symmetry is observed during the quantum nutations.

3. SYMMETRY AND QUANTUM DISCORD

In this section we study the influence of symmetry on the quantum discord between different subsystems (atom - atom, atom - field, field - field). Quantum discord, in quantum information theory, is a measure of nonclassical correlations

between two subsystems. It includes correlations that are due to quantum physical effects but do not necessarily involve quantum entanglement.

The notion of quantum discord (QD) was introduced by Ollivier and Zurek [12, 13] and, independently by L. Henderson and Vlatko Vedral [14]. For simplicity, we have found the QD for several particular cases. Let's start with a system formed of two coupled cavities, in the presence only of one excitation. The possible symmetrical states are $|\psi_1\rangle = [|e_1g_2\rangle + \exp[i\phi]|g_1e_2\rangle]|0,0\rangle/\sqrt{2}$ and $|\psi_2\rangle = [|1,0\rangle + \exp[i\phi]|0,1\rangle]|g_1,g_2\rangle/\sqrt{2}$. The density matrix may be presented through the priory probability on the states $\hat{\rho} = 0.5|\psi_1\rangle\langle\psi_1| + 0.5|\psi_2\rangle\langle\psi_2|$. As both collective states have the same probability, the total entropy of system atom+field, $S = -Sp\{\hat{\rho}\log[\hat{\rho}]\} = 1$, can be easily calculated.

As the density matrix is represented through the diagonal elements relatively to the atomic and field operators, the discord between these subsystems is absent. But the discord can be introduced between the atoms and photons belonging to different cavities. Indeed, calculating the density matrix for atomic and field subsystems, $\rho_A = Sp_F(\rho)$ and $\rho_F = Sp_A(\rho)$, we can estimate the collective entropy between the atoms and photons $S_A = -Sp\{\hat{\rho}_A\log[\hat{\rho}_A]\} = 1$, $S_F = -Sp\{\hat{\rho}_F\log[\hat{\rho}_F]\} = 1$. Considering that first and second atom are two subsystem of the collective atomic states, we may found the density matrix and entropy for each atom $\rho_{A1} = Sp_{A2}(\rho_A)$ and $\rho_{A2} = Sp_{A1}(\rho_A)$, so that the entropies for first and second atom are $S_{A1} = S_{A2} = 2 - 0.75\log(3)$. In this case, quantum mutual information of two atoms is $I = S_{A1} + S_{A2} - S_A = 3 - 1.5\log(3)$.

Let us regard distinguished cavities with a single excitation. The states are described by the expressions: $|\psi_{1c}\rangle = |e_1g_2\rangle|0,0\rangle$; $|\psi_{2c}\rangle = |g_1e_2\rangle]|0,0\rangle$; $|\psi_{3c}\rangle = [|1,0\rangle|g_1,g_2\rangle$; $|\psi_{4c}\rangle = |0,1\rangle|g_1,g_2\rangle$. The quasi classical density matrix, *i.e.*, $\hat{\rho}_c = 0.25|\psi_{1c}\rangle\langle\psi_{1c}| + 0.25|\psi_{2c}\rangle\langle\psi_{2c}| + 0.25|\psi_{3c}\rangle\langle\psi_{3c}| + 0.25|\psi_{4c}\rangle\langle\psi_{4c}|$, is obtained from quantum density matric ρ mediated on the all values of angle ϕ. So, the classical mutual information can be easily calculated. Indeed according to this presentation, the classical entropy the distinguished atoms is $S_{cA} = 1,5$. This result gives the following numerical value for classical mutual information $I_c = S_{cA} - S_{A1} - S_{A2} = 2,5 - 1,5\log(3)$. In this context, the QD for atoms is

$$D = I - I_c = 0,5$$

In the case of two excitations in the cavities we observe the possibilities to found the quantum correlations between the undistinguished atoms and photons Indeed the similar expressions may be obtained for discord between the excitations localized in the first and second cavity

$$
\begin{aligned}
|\psi_{E_p}\rangle &= |g_1g_2\rangle|1_11_2\rangle;\ |\psi_{E_a}\rangle = |e_1e_2\rangle|0_10_2\rangle;\\
|\psi_{AF}\rangle &= \{|e_1g_2\rangle|1_10_2\rangle + \exp[i\phi]|g_1e_2\rangle|0_11_2\rangle\}/\sqrt{2};\\
|\psi_{FA}\rangle &= \{|e_1g_2\rangle|0_11_2\rangle + \exp[i\phi]|g_1e_2\rangle|1_10_2\rangle\}/\sqrt{2};\\
|\psi_{2E_p}\rangle &= |g_1g_2\rangle\{|2_10_2\rangle + \exp[i\phi]|0_12_2\rangle\}/\sqrt{2}
\end{aligned}
$$

We observe that the functions $|\psi_{AF}\rangle$ and $|\psi_{FA}\rangle$ are not factorized relatively to the field and atomic subsystems. The density matrix for a priory probability is given by $\rho = |\psi_{E_a}\rangle\langle|\psi_{E_a}|/5 + |\psi_{E_p}\rangle\langle|\psi_{E_p}|/5 + |\psi_{AF}\rangle\langle\psi_{AF}|/5 + |\psi_{FA}\rangle\langle\psi_{FA}|/5 + |\psi_{2E_p}\rangle\langle|\psi_{2E_p}|/5$.

The entropy for atomic and field subsystem (total entropy $S = -\rho Sp\{\rho\}$) can easily calculated $S = 5(1/5)\log 5 = \log 5$. The entropies for field and atoms take the following values $S_A = \log 5 - (4/5)$ and $S_{Ph} = \log 5 - (2/5)$, so that the quantum mutual information for atoms and field is

$$I = S_A + S_{Ph} - S = \log 5 - 6/5.$$

For quasi classical correlations, we have obtained the following values for entropy $S_c = \log 5 + 3/5$, $S_{cA} = \log 5 - 2/5$ and $S_{cF} = \log 5 + 1/5$. The mutual classical information is

$$I_c = S_{cA} + S_{cF} - S_c = \log 5 - 4/5$$

Now we can estimate the QD for atomic and field subsystems

$$D = I - I_c = 2/5 = 0.4$$

In conclusion we observe that only beginning with two excitations the discord between the field and atoms appears.

4. CONCLUSIONS

In this paper, we have studied the influence of packing geometry of cavities on the possibilities to find the relations between the entangled states and indistinguishable between the particles: atoms or photons. It is demonstrated that the symmetry of packing change drastically the energy spectrum of systems, modifying respectively the quantum behavior of the system. The number of degrees of freedom of the system formed from coupled cavities is reduced as in the Dicke problem for the radiators situated in the volume with the dimension less than the radiation wavelength. In Fig. 2 was presented the distinct properties between different packing geometries of four coupled cavities and their applications in quantum information. The depen-

dence of the quantum discord on the packing symmetry and number of excitations was studied.

5. APPENDIX

In this section we present the coefficients of superposition for eigenstates of interaction part of Hamiltonian. Coefficients for 3 coupled cavities placed in line:

$$\lambda_1 : c[1] = \left(\frac{\lambda_1\lambda_4^2\sqrt{\varepsilon(\varepsilon-\chi)}}{2g^3\varepsilon}, -\frac{(\varepsilon+\chi)\sqrt{\varepsilon(\varepsilon-\chi)}}{2\sqrt{2}g\varepsilon}, \frac{\sqrt{\varepsilon(\varepsilon-\chi)}}{2\varepsilon}, -\frac{\lambda_1(\varepsilon+\chi)\sqrt{\varepsilon(\varepsilon-\chi)}}{2\sqrt{2}g^2\varepsilon} \right),$$

$$\lambda_2 : c[2] = \left(\frac{\lambda_1\lambda_4^2\sqrt{\varepsilon(\varepsilon-\chi)}}{2g^3\varepsilon}, \frac{(\varepsilon+\chi)\sqrt{\varepsilon(\varepsilon-\chi)}}{2\sqrt{2}g\varepsilon}, -\frac{\sqrt{\varepsilon(\varepsilon-\chi)}}{2\varepsilon}, -\frac{\lambda_1(\varepsilon+\chi)\sqrt{\varepsilon(\varepsilon-\chi)}}{2\sqrt{2}g^2\varepsilon} \right),$$

$$\lambda_3 : c[3] = \left(\frac{\lambda_1^2\lambda_4\sqrt{\varepsilon(\varepsilon+\chi)}}{2g^3\varepsilon}, -\frac{(\varepsilon-\chi)\sqrt{\varepsilon(\varepsilon+\chi)}}{2\sqrt{2}g\varepsilon}, -\frac{\sqrt{\varepsilon(\varepsilon+\chi)}}{2\varepsilon}, \frac{\lambda_4(\varepsilon-\chi)\sqrt{\varepsilon(\varepsilon+\chi)}}{2\sqrt{2}g^2\varepsilon} \right),$$

$$\lambda_4 : c[4] = \left(\frac{\lambda_1^2\lambda_4\sqrt{\varepsilon(\varepsilon+\chi)}}{2g^3\varepsilon}, \frac{(\varepsilon-\chi)\sqrt{\varepsilon(\varepsilon+\chi)}}{2\sqrt{2}g\varepsilon}, \frac{\sqrt{\varepsilon(\varepsilon+\chi)}}{2\varepsilon}, \frac{\lambda_4(\varepsilon-\chi)\sqrt{\varepsilon(\varepsilon+\chi)}}{2\sqrt{2}g^2\varepsilon} \right).$$

Four cavities in line:

$$\lambda_1 : c[1] = \left(\frac{\lambda_3\sqrt{\beta(\beta+\chi)}}{2g\beta}, -\frac{\lambda_3\sqrt{\beta(\beta+\chi)}}{2g\beta}, \frac{\sqrt{\beta(\beta+\chi)}}{2\beta}, -\frac{\sqrt{\beta(\beta+\chi)}}{2\beta} \right),$$

$$\lambda_2 : c[2] = \left(\frac{\lambda_1\sqrt{\beta(\beta-\chi)}}{2g\beta}, \frac{\lambda_1\sqrt{\beta(\beta-\chi)}}{2g\beta}, \frac{\sqrt{\beta(\beta-\chi)}}{2\beta}, \frac{\sqrt{\beta(\beta-\chi)}}{2\beta} \right),$$

$$\lambda_3 : c[3] = \left(\frac{\lambda_1\sqrt{\beta(\beta-\chi)}}{2g\beta}, -\frac{\lambda_1\sqrt{\beta(\beta-\chi)}}{2g\beta}, -\frac{\sqrt{\beta(\beta-\chi)}}{2\beta}, \frac{\sqrt{\beta(\beta-\chi)}}{2\beta} \right),$$

$$\lambda_4 : c[4] = \left(\frac{\lambda_3\sqrt{\beta(\beta+\chi)}}{2g\beta}, \frac{\lambda_3\sqrt{\beta(\beta+\chi)}}{2g\beta}, -\frac{\sqrt{\beta(\beta+\chi)}}{2\beta}, -\frac{\sqrt{\beta(\beta+\chi)}}{2\beta} \right),$$

Four cavities in the vertices of a regular tetrahedron:

$$\lambda_1 : c[1] = \left(-\frac{\lambda_2\sqrt{\varepsilon(\varepsilon-3\chi)}}{g\varepsilon\sqrt{2}}, \frac{\sqrt{\varepsilon(\varepsilon-3\chi)}}{\varepsilon\sqrt{2}} \right),$$

$$\lambda_2 : c[2] = \left(\frac{\lambda_1\sqrt{\varepsilon(\varepsilon+3\chi)}}{g\varepsilon\sqrt{2}}, -\frac{\sqrt{\varepsilon(\varepsilon+3\chi)}}{\varepsilon\sqrt{2}} \right),$$

Acknowledgements. This work was supported by the Institutional project 15.817.07F and project for mobility Italy - Moldova 15.820.18.02.04/It.

REFERENCES

1. S. I. Schmid and J. Evers, arXiv:1108.4525 (2011).
2. N. Enaki and S. Bazgan, Physica Scripta **2014**, 014010 (2014).
3. L. M. Duan, M. Lukin, J. I. Cirac and P. Zoller, Nature **414**, 413 (2001).
4. D. Felinto, C. W. Chou, H. Riedmatten, S. V. Polyakov and H. J. Kimble, Phys. Rev. A **72**, 053809 (2005).
5. A. Kuzmich, W. P. Bowen, A. D. Boozer, A. Boca, C. W. Chou, L.-M. Duan and H. J. Kimble, Nature **423**, 731 (2003).

6. R. H. Dicke, Phys. Rev. **93**, 99 (1954).
7. E. Nahmad-Achar, O. Castanos, R. Lopez-Pena and J. G. Hirsch, Phys. Scr. **87**, 038114 (2013).
8. N. Lambert, C. Emary and T. Brandes, Phys. Rev. Lett. **92** (7), 073602 (2004).
9. C. Emary and T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003).
10. N. Enaki, S. Bazgan, Phys. Scr. **2013**, 014022 (2013).
11. A. Gonzalez-Tudela, D. Martin-Cano *et al.*, Phys. Rev. Lett. **106**, 020501 (2011).
12. W. H. Zurek, Annalen der Physik **9**, 855–864 (2000).
13. H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
14. L. Henderson and V. Vedral, J. of Phys. A **34**, 6899 (2001),