A NOTE ON PROPER PROJECTIVE COLLINEATION
IN SPECIAL NON-STATIC SPHERICALLY SYMMETRIC SPACE-TIMES

GHULAM SHABBIR, AMJAD ALI

GIK Institute of Engineering Sciences and Technology, Faculty of Engineering Sciences, Topi,
Swabi, KPK, Pakistan.
E-mail: shabbir@giki.edu.pk

Received May 5, 2014

Abstract. In this paper a study of proper projective collineation in special non-static spherically symmetric space-times is given by using real eigenvalues and eigenbivectors of the Riemann tensor, direct integration and algebraic techniques. From the above study we have shown that when the above space-time admits proper projective collineation they become a very special class of static spherically symmetric space-times.

Key words: real eigenvalues and eigenbivectors, direct integration and algebraic techniques, proper projective collineation.

1. INTRODUCTION

The aim of this paper is to find the existence of proper projective collineation in special non-static spherically symmetric space-times. In this paper an approach, which is given in [1] is used to study the projective collineation for the above space-times. Doubtless, this approach is lengthy but it will definitely tell the existence of a proper projective collineation. Throughout \( M \) represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric \( g \) of signature \((-, +, +, +)\). The curvature tensor associated with \( g_{ab} \), through the Levi-Civita connection, is denoted in component form by \( R_{abcd} \), and the Ricci tensor components are \( R_{ab} = R_{cab} \). The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol \( \mathcal{L} \), respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time \( M \) will be assumed non flat in the sense that the curvature tensor does not vanish over any non empty open subset of \( M \).

Any vector field \( X \) on \( M \) can be decomposed as

\[
X_{a,b} = \frac{1}{2} h_{ab} + F_{ab},
\]

(1)
where \( h_{ab} (= h_{ba}) = L_x g_{ab} \) and \( F_{ab} (= -F_{ba}) \) are symmetric and skew symmetric tensors on \( M \), respectively. Such a vector field \( X \) is called projective if the local diffeomorphisms \( \psi_t \) (for appropriate \( t \)) associated with \( X \) map geodesics into geodesics. This is equivalent to the condition that \( h_{ab} \) satisfies \(^{[2]}\)

\[
h_{ab;c} = 2 g_{ab} \eta_c + g_{ac} \eta_b + g_{bc} \eta_a,
\]

for some smooth closed 1-form on \( M \) with local components \( \eta_a \). Thus \( \eta_a \) is locally gradient because the connection is metric and will, where appropriate, be written as \( \eta_a = \eta_{a,} \) for some function \( \eta \) on some open subset of \( M \). If \( X \) is a projective collineation and \( \eta_{a,b} = 0 \) then \( X \) is called a special projective collineation on \( M \). The statement that \( h_{ab} \) is covariantly constant on \( M \) is, from \(^{(2)}\), equivalent to \( \eta_a \) being zero on \( M \) and is, in turn equivalent to \( X \) being an affine vector field on \( M \) (so that the local diffeomorphisms \( \psi_t \) preserve not only geodesics but also their affine parameters). If \( X \) is projective but not affine then it is called proper projective collineation \(^{[3]}\). The vector field \( X \) is said to be proper special projective collineation, if \( X \) is not affine and \( \eta_{a,b} = 0 \). Further if \( X \) is affine and \( h_{ab} = 2 c g_{ab} \), \( c \in R \) then \( X \) is homothetic (otherwise proper affine). If \( X \) is homothetic and \( c \neq 0 \) it is proper homothetic while if \( c = 0 \) it is Killing.

The second order skew symmetric tensor \( F_{ab} \) is called a bivector (at \( p \)). Regarding \( F_{ab} \) as a skew matrix, its rank is therefore an even number 0, 2 or 4. If it is 0 then \( F_{ab} = 0 \). Suppose if the rank of \( F_{ab} \) is 2 then it is called simple bivector otherwise it is called non-simple (for more details see \(^{[3]}\)). Here, at \( p \in M \) one may choose a orthonormal tetrad \((t, r, \theta, \phi)\) satisfying \(-r^a t_a = r^a t_a = \theta^a \theta_a = \phi^a \phi_a = 1\) (with all others inner products zero). Since at \( p \), the set of bivectors at \( p \) is a six-dimensional vector space which can spanned by the six bivectors given by \(^{[3]}\)

\[
\begin{align*}
1F_{ab} &= 2t(a \phi_b), \quad 2F_{ab} &= 2t(a \phi_1), \quad 3F_{ab} &= 2t(a \phi_2), \\
4F_{ab} &= 2r(a \phi_b), \quad 5F_{ab} &= 2r(a \phi_1), \quad 6F_{ab} &= 2 \theta(a \phi_3).
\end{align*}
\]

In general, however, equation \(^{(2)}\) is difficult to handle directly and alternative techniques are needed. One such technique arises from the following result. Let \( X \) be a projective collineation on \( M \) so that \(^{(1)}\) and \(^{(2)}\) hold and let \( F \) be a real
curvature eigenbivector at \( p \in M \) with eigenvalue \( \lambda \in \mathbb{R} \) (so that \( R_{\alpha\beta}^{\gamma\delta} F_{\gamma\delta} = \lambda F_{\alpha\beta} \) at \( p \)) then at \( p \) one has \([4]\)

\[
P_{ab} F^e b + P_{bc} F^e a = 0 \quad (P_{ab} = \lambda h_{ab} + 2 \eta_{a;b}). \tag{3}
\]

Equation (3) gives a relation between \( F^a b \) and \( P_{ab} \) (a second order symmetric tensor) at \( p \) and reflects the close connection between \( h_{ab} \), \( \eta_{a;b} \) and the algebraic structure of the curvature at \( p \). If \( F \) is simple then the blade of \( F \) (a two dimensional subspace of \( T_p M \)) consists of eigenvectors of \( P \) with same eigenvalue. Similarly, if \( F \) is non-simple then it has two well defined orthogonal timelike and spacelike blades at \( p \) each of which consists of eigenvectors of \( P \) with same eigenvalue but with possibly different eigenvalues for the two blades \([3]\).

2. MAIN RESULTS

Consider a non static spherically symmetric space-time in the usual coordinate system \((t, r, \theta, \phi)\) (labeled by \((x^0, x^1, x^2, x^3)\), respectively) with line element \([5]\)

\[
ds^2 = -e^{4(r,t)} dt^2 + e^{2(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{4}
\]

The above space-time (4) admits three linearly independent Killing vector fields which are

\[
\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \quad \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial \phi}. \tag{5}
\]

The non-zero independent components of the Riemann tensor are

\[
R_{01}^{01} = -\frac{1}{4} e^{-2(r,t)} A(t,r) \equiv \alpha_\ast, \quad R_{12}^{03} = R_{13}^{02} = \frac{1}{2r} B_i(t,r) e^{-B_i(t,r)} = \alpha_i,
\]

\[
R_{23}^{23} = \frac{1}{r^2} (1 - e^{-B_i(t,r)}) \equiv \alpha_4, \quad R_{24}^{12} = R_{25}^{13} = -\frac{1}{2r} B_i(t,r) e^{-B_i(t,r)} = \alpha_i,
\]

\[
R_{25}^{13} = R_{03}^{13} = -\frac{1}{2r} B_i(t,r) e^{-B_i(t,r)} = \alpha_4, \quad R_{12}^{03} = R_{13}^{02} = \frac{1}{2r} B_i(t,r) e^{-B_i(t,r)} = \alpha_i. \tag{6}
\]
where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\alpha_6$ are real functions of $t$ and $r$ only. It also follows that $\alpha_6 = e^{\lambda t} \alpha_5$. One can write the above equation (6) as

$$
R^{ab}_{cd} \ F^{cd} = \alpha_1^1 F^{ab} , \quad R^{ab}_{cd} \ F^{cd} = \alpha_2^2 F^{ab} + \alpha_6^6 F^{ab} , \quad R^{ab}_{cd} \ F^{cd} = \alpha_2^3 F^{ab} + \alpha_6^5 F^{ab} , \quad R^{ab}_{cd} \ F^{cd} = \alpha_2^4 F^{ab} + \alpha_6^4 F^{ab} , \quad R^{ab}_{cd} \ F^{cd} = \alpha_3^5 F^{ab} + \alpha_6^3 F^{ab} , \quad R^{ab}_{cd} \ F^{cd} = \alpha_4^6 F^{ab} .
$$

(7)

It is important to note that the method we are going to follow is given in [1].

Define $W_{ab} = 4 F_{ab} + \lambda^2 F_{ab}$ and $W_{ab} = 5 F_{ab} + \eta^3 F_{ab}$. We are interested to find that whether these bivectors $W_{ab}$ and $W_{ab}$ are the real eigenbivectors of the Riemann tensor for the real choice of $\lambda$ and $\eta$. Equation (7) gives

$$
R^{ab}_{cd} \ W^{cd} = (\alpha_3 + \lambda \alpha_6) \left( 4 F^{ab} + \left( \frac{\alpha_5 + \lambda \alpha_5}{\alpha_5 + \eta \alpha_5} \right)^2 F^{ab} \right)
$$

and

$$
R^{ab}_{cd} \ W^{cd} = (\alpha_3 + \lambda \alpha_6) \left( 5 F^{ab} + \left( \frac{\alpha_5 + \eta \alpha_5}{\alpha_5 + \eta \alpha_5} \right)^3 F^{ab} \right).
$$

$W_{ab}$ and $W_{ab}$ are the eigenbivectors of the Riemann tensor for the choice of $\lambda_1 = \eta_1 = \frac{(\alpha_2 - \alpha_3) + \rho}{2\alpha_6}$ and $\lambda_2 = \eta_2 = \frac{(\alpha_2 - \alpha_3) - \rho}{2\alpha_6}$, where $\rho^2 = (\alpha_3 - \alpha_2)^2 - 4\alpha_5\alpha_6$ and $\alpha_6 \neq 0$. The case when $\alpha_6 = 0$ will be discussed later. One can easily see that in general $\lambda_1$, $\eta_1$, $\lambda_2$ and $\eta_2$ are not real. Hence the eigenbivectors of the Riemann tensor are not real in general. Here, we are only interested in real eigenbivectors of the Riemann tensor with real eigenvalues. There exist the following possibilities $\rho^2 \geq 0$ or $(\alpha_3 - \alpha_2)^2 - 4\alpha_5\alpha_6 \geq 0$. Substituting the above information back we get

$$
R^{ab}_{cd} \left( 4 F^{cd} + \lambda_1^2 F^{cd} \right) = \left( \alpha_3 + \alpha_2 + \rho \right) \left( 4 F^{ab} + \lambda_1^2 F^{ab} \right) ,
$$

$$
R^{ab}_{cd} \left( 5 F^{cd} + \lambda_2^3 F^{cd} \right) = \left( \alpha_3 + \alpha_2 - \rho \right) \left( 5 F^{ab} + \lambda_2^3 F^{ab} \right).
$$

The bivectors $W_{ab} = (4 F_{ab} + \lambda^2 F_{ab})$ and $W_{ab} = (5 F_{ab} + \lambda^3 F_{ab})$ are simple. Now define $\gamma_1 = \frac{1}{2} (\alpha_3 + \alpha_2 + \rho)$ and $\gamma_2 = \frac{1}{2} (\alpha_3 + \alpha_2 - \rho)$. Here, at $p \in M$ one can choose the tetrad $(t, r, \theta, \phi)$ satisfying $-t^a t_a = r^a r_a = \theta^a \theta_a = \phi^a \phi_a = 1$ (with
all other inner products zero). Here the vector fields are chosen as $t_a = e^2 \delta^0_a$, $r_a = e^2 \delta^1_a$, $\theta_a = r \delta^2_a$ and $\phi_a = r \sin \theta \delta^3_a$. It is important to note that we are using $(t, r, \theta, \phi)$ as both coordinates and vector fields. The eigenvectors $^1W_{ab}$ and $^2W_{ab}$ of the curvature tensor at $p \in M$ are simple with blades spanned by the vector pairs $(r + \lambda_1 t, \theta)$ each with eigenvalue $\gamma_1(p)$ and $(r + \lambda_2 t, \phi)$ each with eigenvalue $\gamma_2(p)$. We consider the open sub region where $\gamma_1$ and $\gamma_1$ are nowhere equal. The rest will be considered later. It is important to note that we are considering the case when $\rho^2 > 0$. The case when $\rho = 0 \Rightarrow \gamma_1 = \gamma_2$ which gives contradiction to our assumption that $\gamma_1$ and $\gamma_1$ are nowhere equal. The case when $\gamma_1 = \gamma_2$ will consider later. At $p$, the second order symmetric tensors is a 10-dimensional vector space which can spanned by the 10 basis symmetric tensors given by: $L_a L_b$, $S_a S_b$, $\theta_a \theta_b$, $\phi_a \phi_b$, $2L_a S_b$, $2L_a \theta_b$, $2S_a \theta_b$, $2S_a \phi_b$ and $2\theta_a \phi_b$, where $L_a \equiv r_a + \lambda_1 t_a$ and $S_a \equiv r_a + \lambda_2 t_a$. It also follows that $L_a S^a = 1 - \lambda_1 \lambda_2$, $L_a L^a = 1 - \lambda_1^2$, $S_a S^a = 1 - \lambda_2^2$ and $\lambda_1$ and $\lambda_2$ are nowhere equal. Now at $p$, the symmetric tensor $P_{ab} (= \gamma_1 h_{ab} + 2 \eta_{a,b})$ can be written as a linear combination of the basis members of the 10-dimensional vector space and use the fact that $P_{ab} = \gamma_1 h_{ab} + 2 \eta_{a,b}$ has eigenvectors $r + \lambda_1 t, \theta$ with same eigenvalue, say $\zeta_1$ and similarly for the symmetric tensor $P_{ab} (= \gamma_2 h_{ab} + 2 \eta_{a,b})$ can be written as a linear combination of the basis members of the 10-dimensional vector space and use the fact that $P_{ab} = \gamma_2 h_{ab} + 2 \eta_{a,b}$ has eigenvectors $r + \lambda_2 t, \phi$ with same eigenvalue, say $\zeta_2$. Hence on $M$ one has after the use of completeness relation ($g_{ab} = \gamma_3 L_a L_b + \gamma_3 S_a S_b + \theta_a \theta_b + \phi_a \phi_b$)

$$
\begin{align*}
\gamma_1 h_{ab} + 2 \eta_{a,b} &= \zeta_1 g_{ab} + a_1 S_a S_b + b_1 \phi_a \phi_b + c_1 \theta_a \theta_b + c_1 S_a \phi_b + c_1 S_b \phi_a + c_1 \theta_a \phi_b + c_1 \theta_b \phi_a,
\gamma_2 h_{ab} + 2 \eta_{a,b} &= \zeta_2 g_{ab} + a_2 L_a L_b + b_2 \theta_a \theta_b + c_2 L_a \theta_b + c_2 L_b \theta_a + c_2 \theta_a \theta_b + c_2 \theta_b \theta_a,
\end{align*}
$$

(8)

where $a_1, a_2, b_1, b_2, c_1$ and $c_2$ are real functions on $M$. Since $\gamma_1 \neq \gamma_2$ equation (8) gives

$$
\begin{align*}
h_{ab} &= a_1 g_{ab} + a_2 S_a S_b + a_4 L_a L_b + a_6 \theta_a \theta_b + a_7 \phi_a \phi_b + a_8 \theta_a \phi_b + 2a_8 S_a \phi_b + 2a_8 L_a \theta_b,
\eta_{a,b} &= b_1 g_{ab} + b_2 S_a S_b + b_4 L_a L_b + b_5 \theta_a \theta_b + b_6 \phi_a \phi_b + b_7 \theta_a \phi_b + 2b_8 S_a \phi_b + 2b_8 L_a \theta_b,
\end{align*}
$$

(9)

where $a_1, a_2, a_4, a_6, a_7, a_8, b_1, b_2, b_4, b_5, b_6, b_7, b_8$ and $b_9$ are functions on some open subregion of $M$. Now we are interested in finding the projective vector fields by using the relation
Writing equation (10) explicitly and using first equation of (9) and (4) we get ten coupled non-linear equations. In order to find the projective vector field we need to solve ten equations. After some tedious and lengthy calculation one finds that \( \eta_a = 0 \). Hence no proper projective collineations exist in this case. Projective collineations in this case are Killing vector fields.

Now consider the case when \( \gamma_1 = \gamma_2 \). Equation \( \gamma_1 = \gamma_2 \Rightarrow \rho = 0 \). Substituting back and again writing equation (10) explicitly in to ten equations. After some lengthy calculation one finds that \( \eta_a = 0 \) which implies in this case no proper projective collineations exists. Projective collineations in this case are Killing vector fields.

Now consider the case when \( \alpha_6 = 0 \Rightarrow B(t, r) = 0 \Rightarrow B = B(r) \) and also we have \( \alpha_5 = 0 \). It is important to mention here that throughout in this paper we have \( B = B(r) \). Substituting \( \alpha_6 = 0 \) and \( \alpha_5 = 0 \) in equation (7) we get

\[
\begin{align*}
R_{cd}^{ab}1 F_{cd} &= \alpha_1^{1} F_{ab}, & R_{cd}^{ab}2 F_{cd} &= \alpha_2^{2} F_{ab}, \\
R_{cd}^{ab}3 F_{cd} &= \alpha_3^{3} F_{ab}, & R_{cd}^{ab}4 F_{cd} &= \alpha_3^{4} F_{ab}, \\
R_{cd}^{ab}5 F_{cd} &= \alpha_5^{5} F_{ab}, & R_{cd}^{ab}6 F_{cd} &= \alpha_5^{6} F_{ab}.
\end{align*}
\]

Here, at \( p \in M \) one can choose the tetrad \((t, r, \theta, \phi)\) satisfying

\[
t = e^\Delta \delta^a, \quad r = e^\Delta \delta^a, \quad \theta = r \delta^2, \quad \phi = r \sin \theta \delta^3.
\]

We are considering the open sub region where \( \alpha_2 \) and \( \alpha_5 \) are nowhere equal and \( \alpha_2 \neq 0 \). The rest will be considered latter. It is important to note that we are using \((t, r, \theta, \phi)\) as both coordinates and vector fields. Thus at \( p \), the tensor \( P_{ab} = \alpha_2 h_{ab} + 2 \eta_{a,b} \) has eigenvectors \( t, \theta, \phi \) with same eigenvalue, say \( \beta_1 \), and \( P_{ab} = \alpha_5 h_{ab} + 2 \eta_{a,b} \) has eigenvectors \( r, \theta, \phi \) with same eigenvalue, say \( \beta_2 \). Hence on \( M \) one has after the use of completeness relation

\[
\begin{align*}
\alpha_2 h_{ab} + 2\eta_{a,b} &= \beta_1 g_{ab} + \beta_3 r_a r_b, \\
\alpha_5 h_{ab} + 2\eta_{a,b} &= \beta_5 g_{ab} + \beta_5 r_a r_b.
\end{align*}
\]
where $\beta_1, \beta_2, \beta_3$, and $\beta_4$ are real functions on $M$. Since $\alpha_2 \neq \alpha_3$, then it follows from (6) that

$$h_{ab} = Q g_{ab} + Dr_a r_b + E t_a t_b, \quad \eta_{a;b} = F g_{ab} + Gr_a r_b + K t_a t_b,$$  \hspace{1cm} (12)$$

where $D, E, F, G, K$ and $Q$ are functions on some open subregion of $M$. Next one substitutes the first equation of (12) in (2), we get

$$Q_c g_{ab} + D_c r_a + D r_a r_b + D_{b;r} r_a + E_c t_a t_b + E t_a t_b =$$

$$= 2 g_{ab} \eta_c + g_{ac} \eta_b + g_{bc} \eta_a.$$ \hspace{1cm} (13)

Contracting the above equation with $\theta^a \phi^b$, and then comparing both sides, we have $\eta_a \theta^a = \eta_a \phi^a = 0$ which implies $\eta = \eta(t, r)$. Now contracting equation (13) with $\theta^a \theta^b$ we get $Q_c = 2 \eta_c$ which implies $Q = Q(t, r)$. Once again contracting equation (13) with $t^a t^b$ and $r^a r^b$, we get $E = E(t)$ and $D = D(r)$, respectively. Now consider the first equation of (12) and using (4), we get the following non-zero components of $h_{ab}$

$$h_{00} = [E(t) - Q(t, r)] e^{\theta(t)}, \quad h_{11} = [Q(t, r) + D(r)] e^{\theta(r)},$$

$$h_{22} = Q(t, r) r^2, \quad h_{33} = Q(t, r) r^2 \sin^2 \theta.$$ \hspace{1cm} (14)

Now we are interested in finding the projective vector fields by using the relation (10). Writing equation (10) explicitly and using (4) and (14) we get

$$A_t (t, r) X^0 + A_r (t, r) X^1 + 2 X^0 = Q(t, r) - E(t)$$ \hspace{1cm} (15)

$$e^{\theta(t)} X^0 + e^{\theta(t)} X^0 = 0$$ \hspace{1cm} (16)

$$r^2 X^2_0 - e^{\theta(t)} X^0 = 0$$ \hspace{1cm} (17)

$$r^2 \sin^2 \theta X^3_0 - e^{\theta(t)} X^0 = 0$$ \hspace{1cm} (18)

$$B_r (r) X^1 + 2 X^1 = Q(t, r) + D(r)$$ \hspace{1cm} (19)

$$r^2 X^1 + e^{\theta(r)} X^1 = 0$$ \hspace{1cm} (20)

$$r^2 \sin^2 \theta X^3_1 + e^{\theta(r)} X^1 = 0$$ \hspace{1cm} (21)

$$2 X^1 + 2r X^2_2 = r Q(t, r)$$ \hspace{1cm} (22)

$$\sin^2 \theta X^3_1 + X^3_1 = 0$$ \hspace{1cm} (23)
\[ 2X^1 + 2r \cot \theta X^2 + 2r X^3 = r Q(t, r). \]  \hfill (24)

Considering equations (17) and (18) and differentiating with respect to \( \phi \) and \( \theta \), respectively and subtracting them we get

\[ - \left[ \sin^2 \theta X^3 \right]_{r,t} + X^2_{03} = 0. \]  \hfill (25)

Differentiating equation (23) with respect to \( t \) we get

\[ \sin^2 \theta X^3_{02} + X^2_{03} = 0. \]  \hfill (26)

Subtracting equation (25) from equation (26) and upon integrating we get

\[ X^3 = \csc \theta \int E^1(t, r, \phi) dt + E^2(r, \theta, \phi), \]

where \( E^1(t, r, \phi) \) and \( E^2(r, \theta, \phi) \) are functions of integration. Using the above information in equation (18) one has

\[ X^0 = e^{-A(r, \theta)} r^2 \sin \theta \int E^1(t, r, \phi) d\phi + E^3(t, r, \theta), \]

where \( E^3(t, r, \theta) \) is a function of integration. Substituting the value of \( X^0 \) in equations (16) and (17) we get

\[
\begin{align*}
X^1 &= r^2 \sin \theta e^{-B(r)} \left[ \int E^1(t, r, \phi) d\phi \right] dt + 2r \sin \theta e^{-B(r)} \left[ \int E^1(t, r, \phi) d\phi \right] \frac{dA}{dt} - r^3 \sin \theta e^{-B(r)} \int \left[ A \int E^1(t, r, \phi) dt \right] d\phi + \int e^{d(r, \theta)} E^1_j(t, r, \theta) dt + E^4(r, \theta, \phi), \\
X^2 &= \cos \theta \int E^1(t, r, \phi) d\phi dt + \frac{1}{r^2} \int e^{d(r, \theta)} E^3_j(t, r, \theta) dt + E^5(r, \theta, \phi),
\end{align*}
\]

where \( E^4(r, \theta, \phi) \) and \( E^5(r, \theta, \phi) \) are functions of integration. In order to find the projective vector field \( X \) we are interested to find the all unknown functions \( E^1(t, r, \phi), E^2(r, \theta, \phi), E^3(t, r, \theta), E^4(r, \theta, \phi) \) and \( E^5(r, \theta, \phi) \). To avoid lengthy details we shall write only the results. From the lengthy and tedious calculations there exists only one case when the above space-time admits proper projective collineation which is in this case \( A = \ln \left( b r^2 \right) \) and \( B = \ln \left( \frac{c}{b r^2 + a} \right) \),

where \( a, b, c \in \mathbb{R} (b \neq 0, c \neq 0) \). The space-time (4), after a rescaling of \( t \), takes the form

\[ ds^2 = -r^2 dt^2 + \frac{c}{br^2 + a} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]  \hfill (27)
The above space-time (27) admits four linearly independent Killing vector fields which are
\[
\phi_t, \cos \phi \phi_{\theta}, \sin \phi \phi_{\theta}, \sin \phi \phi_{\phi}, \cos \phi \phi_{\phi}.
\]
Proper projective collineation after subtracting Killing vector fields is
\[
X = (0, \frac{r}{2}(br^2 + a), 0, 0)
\]
and one form is \( \eta_a = (rb) r_a \). The above space-time (27) becomes special class of static spherically symmetric space-time.

Now consider the case when \( \alpha_2 = 0 \). From equation (6) one can see that the rank of the \( 6 \times 6 \) Riemann tensor is 3 or \( R^a_{\ bcd} t^d = 0 \), where \( t^a \) is a timelike vector field and unique solution of \( R^a_{\ bcd} t^d = 0 \). Here, it is important to mention here that \( B = B(r) \) and \( \alpha_2 \neq \alpha_3 \). The condition \( \alpha_2 = 0 \Rightarrow A(t, r) = 0 \) which gives \( A = A(t) \). The line element (4) can, after a rescaling of \( t \), be written in this form
\[
ds^2 = -dt^2 + e^{\alpha_2 / \alpha_3} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]
The above space-time is 1+3 decomposable. It follows from [3] that space-time (29) does not admit proper projective collineation. The projective collineation admitted by (29) is a proper affine vector field which is \( t t^a \).

Consider when \( \alpha_2 = \alpha_3 \) (and excluding the special case when \( A = \text{constant and } B = \text{constant} \neq 0 \)) in (4). It follows from [3, 6] that projective collineation admitted by (4) are Killing vector fields which are given in equation (5).

Now consider the special case when \( A = \text{constant and } B = \text{constant} \neq 0 \). The rank of the \( 6 \times 6 \) Riemann tensor is 1 and there exists two independent solutions, which are \( R^a_{\ bcd} t^d = 0 \) and \( R^a_{\ bcd} r^d = 0 \), but only one independent covariantly constant vector field \( t_a = t_a \) satisfying \( t_a; b = 0 \). Substituting the above information in equation (4) and after a rescaling of \( t \), the line element takes the form
\[
ds^2 = -dt^2 + kdr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]
where \( k(= e^\xi) \in \mathbb{R}(k \neq 0 \text{ or } 1) \). The space time is clearly 1+3 decomposable but the rank of the \( 6 \times 6 \) Riemann tensor is 1. It follows from [3] that the above space-time (30) which admits proper special projective collineation which is:
\[
U = (r^2, tr^2, 0, 0).
\]
3. CONCLUSIONS

A study of special non-static spherically symmetric space-times according to their proper projective symmetry is given by using the direct integration and algebraic techniques and real eigenvalues and eigenvectors. Using the above mentioned techniques we have proved that the special class of the above space-times (4) (which become the special class of static spherically symmetric space-times) admit proper projective collineation. This is the space-time given in equation (27) and proper projective collineation is given in equation (28). It is important to mention that different approaches [7–33] were adopted to study projective collineations.

REFERENCES

7. A. Barnes, Class. Quantum Grav. 10, 1139 (1993).
8. G. S. Hall and D. P. Lonie, Class. Quantum Grav. 28, 083101 (2011).
12. G. S. Hall and D. P. Lonie, Class. Quantum Grav. 24, 3617 (2007).
15. G. S. Hall and D. P. Lonie, Class. Quantum Grav. 26, 125009 (2009).
16. G. S. Hall and D. P. Lonie, Class. Quantum Grav. 21, 4549 (2004).
17. G. Shabbir, Class. Quantum Grav. 21, 339 (2004).