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TOPOLOGICAL SOLITONS AND OTHER SOLUTIONS TO POTENTIAL KORTEWEG-DE VRIES EQUATION

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Abstract. This paper studies the potential Korteweg-de Vries equation. The topological soliton solution is obtained by the aid of ansatz method. The mapping method reveals a list of solutions that include cnoidal waves, snoidal waves and singular solutions in three limiting cases. The Lie symmetry analysis is carried out to obtain several other solutions. Finally, a numerical simulation of the topological soliton is also given in this paper.

Key words: potential Korteweg-de Vries equation, topological soliton solution.

1. INTRODUCTION

The study of nonlinear evolution equations (NLEEs) is going on for the past few decades and it is still burning bright [1-25]. There exits a plethora of integration tools that solves these NLEEs thus revealing abundance of solutions. These solutions turn out to be extremely useful in the areas of Applied Mathematics, Theoretical Physics and Engineering Sciences. Therefore, it is imperative to dig this area even deeper in order to find something exciting. This paper is going to shine some light on one such NLEE that is the potential Korteweg-de Vries (p-KdV) equation.

There are several integration tools that are available to address NLEEs. Some of these commonly studies techniques are the variational iteration method, traveling wave hypothesis, G'/G-expansion method, exp-function method, simplest equation method, Adomian decomposition method and many more. The results that are obtained by applying these techniques are essentially equivalent. This paper will address three analytical methods to extract closed form solutions to the pKdV equation. These are the ansatz method, mapping method and the Lie symmetry analysis. Finally, the numerical simulation will be given in order to support the analytical result.

2. TOPOLOGICAL SOLITON SOLUTION

The dimensionless form of the potential Korteweg-de Vries (p-KdV) equation that is going to be studied in this paper is given by [23–25]:

$$q_t + a\left(q_x\right)^2 + bq_{xxx} = 0, \tag{1}$$

where in Eq. (1), $q(x,t)$ is the dependent variable, while x and t are the independent variables. The parameters a and b are real valued constants. Equation (1) arises in the study of water waves where the first term is the evolution term, while the coefficient of a is the nonlinear term and finally the coefficient of b is the dispersion term. This section is going to focus on obtaining the topological 1-soliton solution of Eq. (1) by the *ansatz method*. These topological soliton solutions are also known as the *shock waves* or *kinks*. In order to start off, the initial hypothesis is taken to be

$$q(x,t) = A\tanh^p[B(x-vt)], \tag{2}$$

where A and B are free parameters while v is the velocity of the soliton. The value of the unknown exponent p will fall out during the course of derivation of the solutions to Eq. (1). Now, substituting Eq. (2) into Eq. (1) gives

$$\begin{aligned}
&-pvAB\left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right) + \\
&+ap^2A^2B^2\left(\tanh^{2p-2}\tau - 2\tanh^{2p}\tau + \tanh^{2p+2}\tau\right) + \\
&+bpAB^3\left[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2 + (p-1)(p-2)\right\}\tanh^{p-1}\tau + \right.\\
&\left.+\left\{2p^2 + (p+1)((p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\right] = 0,
\end{aligned} \tag{3}$$

where the notation

$$\tau = B(x-vt) \tag{4}$$

was adopted. From Eq. (3), equating the exponents $2p$ and $p+1$ gives

$$p = 1. \tag{5}$$

It needs to be noted that the same value of p is obtained when the exponent pairs $2p-2$ and $p-1$ or $2p+2$ and $p+3$ are equated against each other. Therefore, from Eq. (3), setting the coefficients of the linearly independent functions $p+j$ for $j=-1,1$ and 3 to zero leads to

$$v=2aAB-8bB^2,\ v=aAB-2bB^2,\ A=\frac{6bB}{a} \qquad (6, 7, 8)$$

It must now be noted that on equating the two values of the velocity v of the soliton from (6) and (7) also leads to (8) which is the relation between the free parameters and the coefficients of dispersion and nonlinearity. This therefore implies that the system of soliton parameters is closed for the p-KdV equation. Also, Eq. (8) prompts the condition

$$a\neq 0\,. \qquad (9)$$

From Eq. (3), it can be easily observed that besides the linearly independent functions as indicated above, there is a stand-alone linearly independent function, namely the coefficient of $\tanh^{p-3}\tau$. The coefficient of this term also implies the same value of the unknown exponent as seen in Eq. (5). Also, from this term $p\neq 2$, since this would lead to inconsistencies. Hence, finally, the topological 1-soliton solution to Eq. (1) is given by

$$q(x,t)=A\tanh[B(x-vt)]\,, \qquad (10)$$

where the free parameters A and B are connected as in Eq. (8) while the velocity of the soliton is given by Eq. (6) or Eq. (7).

3. MAPPING METHOD

Here, we assume the travelling wave solution of Eq. (1) in the form of Eq. (4) so that Eq. (1) reduces to the ordinary differential equation

$$-vq_\tau+aBq_\tau^2+B^2q_{\tau\tau\tau}=0. \qquad (11)$$

By using the substitution $q_\tau=Q$ in Eq. (11), we get

$$-vQ+aBQ^2+B^2Q_{\tau\tau}=0. \qquad (12)$$

We rewrite Eq. (12) in the form

$$EQ_{\tau\tau}+FQ+GQ^2=0 \qquad (13)$$

where E,F and G are given by

$$E=B^2,\quad F=-v,\quad G=aB. \qquad (14)$$

By following the *mapping method* [1, 2], one can easily see that a solution to Eq. (13) will be in the form

$$Q=A_0+A_1 f, \tag{15}$$

where f satisfies a set of equations

$$f''=p+sf+rf^2,\ f'^2=2pf+sf^2+\frac{2}{3}rf^3, \tag{16}$$

and the coefficients $A_0, A_1\, p, s$ and r will all be determined. Equation (15) gives the mapping relation between solutions to Eqs. (13) and (16).

Substituting Eqs. (15) and (16) into Eq. (13) and equating the coefficients of powers of f, we obtain

$$EA_1 r+GA_1^2=0, \tag{17}$$

$$EA_1 s+FA_1+2GA_0 A_1=0, \tag{18}$$

$$EA_1 p+FA_0+GA_0^2=0. \tag{19}$$

From Eqs. (17) and (18), we obtain

$$A_0=-\frac{Es+F}{2G},\ A_1=-\frac{Er}{G}. \tag{20}$$

Equation (19) gives rise to the constraint relation

$$v^2=B^4(s^2-4pr). \tag{21}$$

Case 1. $p=2,\ s=-4(1+m^2),\ r=6m^2$

Equation (16) has the solution $f(\tau)=\mathrm{sn}^2(\tau)$. So, we obtain the periodic wave solution (PWS) of Eq. (13) as

$$Q(\tau)=\frac{4B^2(1+m^2)+v}{2aB}-\frac{6m^2B}{a}\mathrm{sn}^2(\tau). \tag{22}$$

When $m\to 1$, Eq. (22) reduces to the solution

$$Q(\tau)=\frac{8B^2+v}{2aB}-\frac{6B}{a}\tanh^2(\tau). \tag{23}$$

Integrating Eq. (23) with respect to τ, we obtain the solution of Eq. (11) as

$$q(\tau)=\frac{v-4B^2}{2aB}\tau+\frac{6B}{a}\tanh(\tau). \tag{24}$$

So, the solution of Eq. (1) can be written as

$$q(x,t)=\frac{v-4B^2}{2aB}\big(B(x-vt)\big)+\frac{6B}{a}\tanh\big(B(x-vt)\big). \tag{25}$$

Since this solution is unbounded because of the first term in Eq. (25), we set its coefficient zero with $v=4B^2$ which is the velocity of the *topological soliton*.
Thus we get the solution of Eq. (1) as

$$q(x,t)=\frac{6B}{a}\tanh\big(B(x-4B^2t)\big). \tag{26}$$

Case 2. $p=2(1-m^2),\ s=4(2m^2-1),\ r=-6m^2$

Equation (16) has the solution $f(\tau)=\text{cn}^2(\tau)$. So, we obtain the PWS of Eq. (13) as

$$Q(\tau)=-\frac{4B^2(2m^2-1)-v}{2aB}+\frac{6m^2B}{a}\text{cn}^2(\tau). \tag{27}$$

When $m\to 1$, Eq. (27) reduces to the solution

$$Q(\tau)=\frac{v-4B^2}{2aB}+\frac{6B}{a}\text{sech}^2(\tau). \tag{28}$$

Integrating Eq. (28) with respect to τ and looking for a bounded solution as in case 1, we get the same topological soliton solution (26).

Case 3. $p=-2(1-m^2),\ s=4(2-m^2),\ r=-6$

Equation (16) has the solution $f(\tau)=\text{dn}^2(\tau)$. So, we obtain the PWS of Eq. (13) as

$$Q(\tau)=-\frac{4B^2(2-m^2)-v}{2aB}+\frac{6B}{a}\text{dn}^2(\tau). \tag{29}$$

When $m\to 1$, Eq. (29) reduces to the same solution (28) and will generate the same topological soliton (26).

Case 4. $p=2(1-m^2),\ s=4(2-m^2),\ r=6$.

Equation (16) has the solution $f(\tau)=\text{cs}^2(\tau)$. So, we obtain the PWS of Eq. (13) as

$$Q(\tau)=-\frac{4B^2(2-m^2)-v}{2aB}-\frac{6B}{a}\text{cs}^2(\tau). \tag{30}$$

When $m\to 1$, Eq. (30) reduces to the solution

$$Q(\tau)=\frac{v-4B^2}{2aB}-\frac{6B}{a}\text{csch}^2(\tau). \tag{31}$$

Integrating Eq. (31) with respect to τ and setting $v=4B^2$, we get the *singular solution*

$$q(x,t)=\frac{6B}{a}\coth\left(B(x-4B^2t)\right). \tag{32}$$

Case 5. $p=-2m^2(1-m^2)$, $s=4(2-m^2)$, $r=6$

Equation (16) has the solution $f(\tau)=\text{ds}^2(\tau)$. So, we obtain the PWS of Eq. (13) as

$$Q(\tau)=-\frac{4B^2(2m^2-1)-v}{2aB}-\frac{6B}{a}\text{ds}^2(\tau). \tag{33}$$

In the limiting case, the Eq. (33) gives rise to the same singular solution (32).

4. MODIFIED MAPPING METHOD

Now, we use the *modified mapping method* [3, 4] to find some exact solutions of Eq. (1). In this case, we assume the solution of Eq. (13) in the form

$$Q=B_1 f^{-1}+A_0+A_1 f, \tag{34}$$

where A_i and B_i are constants to be determined and f satisfies Eq. (16).

Equation (34) gives an algebraic mapping relation between solutions to Eqs. (13) and (16). Substituting Eqs. (34) and (16) into Eq. (13) and equating the coefficients of powers of f, we obtain

$$EA_1 r+GA_1^2=0, \tag{35}$$

$$EA_1 s+FA_1+2GA_0A_1=0, \tag{36}$$

$$EA_1 p+\frac{1}{3}EB_1 r+FA_0+GA_0^2+2GA_1B_1=0, \tag{37}$$

$$EB_1 s+FB_1+2GA_0B_1=0, \tag{38}$$

$$3pEB_1+GB_1^2=0. \tag{39}$$

From Eqs. (36) and (38), we get

$$A_0=-\frac{Es+F}{2G}. \tag{40}$$

Equations (35) and (39) give

$$A_1=-\frac{Er}{G},\ B_1=-\frac{3pE}{G}. \tag{41}$$

Equation (37) gives rise to the constraint relation

$$v^2=B^4(s^2+16pr). \tag{42}$$

with the constraint relation (42), we have only three cases to consider.

Case 1. $p = 2,\ s = -4(1+m^2),\ r = 6m^2$

Equation (16) has the solution $f(\tau)=\text{sn}^2(\tau)$. So, we obtain the PWS of eq. (13) as

$$Q(\tau)=\frac{4B^2(1+m^2)+v}{2aB}-\frac{6m^2B}{a}\text{sn}^2(\tau)-\frac{6B}{a}\text{ns}^2(\tau). \tag{43}$$

When $m\to 1$, eq. (43) reduces to the solution

$$Q(\tau)=\frac{8B^2+v}{2aB}-\frac{6B}{a}\tanh^2(\tau)-\frac{6B}{a}\coth^2(\tau). \tag{44}$$

Integrating eq. (44) with respect to τ, we obtain the solution of eq. (11) as

$$q(\tau)=\frac{v-16B^2}{2aB}\tau+\frac{6B}{a}\tanh(\tau)+\frac{6B}{a}\coth(\tau). \tag{45}$$

Setting $v=16B^2$, we get the singular solution of eq. (1) as

$$q(x,t)=\frac{6B}{a}\tanh\left(B(x-16B^2t)\right)+\frac{6B}{a}\coth\left(B(x-16B^2t)\right). \tag{46}$$

Case 2. $p = -2(1-m^2),\ s = 4(2-m^2),\ r = -6$

Equation (16) has the solution $f(\tau)=\text{dn}^2(\tau)$. So, we obtain the PWS of Eq. (13) as

$$Q(\tau)=-\frac{4B^2(2-m^2)-v}{2aB}+\frac{6B}{a}\text{dn}^2(\tau)+\frac{6(1-m^2)B}{a}\text{nd}^2(\tau). \tag{47}$$

when $m\to 1$, Eq. (47) reduces to the solution (28) and will generate the topological soliton (26).

Case 3. $p = 2,\ s = 4(2-m^2),\ r = 6(1-m^2)$

Equation (16) has the solution $f(\tau)=\text{sc}^2(\tau)$. So, we obtain the PWS of Eq. (13) as

$$Q(\tau)=-\frac{4B^2(2-m^2)-v}{2aB}-\frac{6B(1-m^2)}{a}\mathrm{sc}^2(\tau)-\frac{6B}{a}\mathrm{cs}^2(\tau). \quad (48)$$

when $m\to 1$, Eq. (48) reduces to the solution (31) and will generate the singular solution (32).

5. LIE SYMMETRY ANALYSIS

In this section, we will apply the *Lie group method* [3, 15, 16], sometimes also called symmetry analysis, on p-KdV equation (1). Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. Some of the recent contributions in this field are [14, 17].

First, let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x^* &\to x+\varepsilon\xi(x,t,q),\\ t^* &\to t+\varepsilon\tau(x,t,q),\\ q^* &\to q+\varepsilon\eta(x,t,q), \end{aligned} \quad (49)$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V=\xi(x,t,q)\frac{\partial}{\partial x}+\tau(x,t,q)\frac{\partial}{\partial t}+\eta(x,t,q)\frac{\partial}{\partial q}. \quad (50)$$

The symmetry group of equation (1) will be generated by the vector field of the form (50). Applying the third prolongation $\mathrm{pr}^{(3)}V$ of V to equation (1), we find that the coefficient functions ξ,τ and η must satisfy the symmetry condition

$$\eta^t+2aq_x\eta^x+b\eta^{xxx}=0, \quad (51)$$

where η^t,η^x and η^{xxx} are infinitesimals [3, 16].

We substitute the values of η^t,η^x and η^{xxx} in Eq. (51) and we replace q_t by (1). On substituting the coefficients of different differentials equal to zero leads to the system of determining equations.

Solving this system of determining equations provides the following forms for the infinitesimal elements ξ,τ and η:

$$\eta = \frac{x}{2a}a_3 - \frac{q}{3}a_4 + a_5$$
$$\xi = a_2 + ta_3 + \frac{x}{3}a_4 \quad (52)$$
$$\tau = ta_4 + a_1,$$

where a_1, a_2, a_3, a_4 and a_5 are arbitrary constants.

The corresponding vector fields are given as

$$V_1 = \frac{\partial}{\partial t},\ V_2 = \frac{\partial}{\partial x},\ V_3 = t\frac{\partial}{\partial x} + \frac{x}{2a}\frac{\partial}{\partial q},\ V_4 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{q}{3}\frac{\partial}{\partial q},\ V_5 = \frac{\partial}{\partial q}. \quad (53)$$

5.1. SYMMETRY REDUCTIONS AND EXACT SOLUTIONS

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In this section, we will use the symmetries calculated in the previous subsection to obtain exact solutions of Eq. (1).

To obtain the symmetry reductions of Eq. (1), we have to solve the characteristic equation

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}q}{\eta}, \quad (54)$$

where ξ, τ and η are given by Eq. (52). To solve Eq. (54), we will consider the following cases:

(i) $V_3 + \mu V_4$, (ii) $V_5 + \alpha V_2 + \beta V_1$, (iii) $V_1 + \gamma V_2$.

5.1.1. Vector Field $V_3 + \mu V_4$

Solving the characteristic equation (54), we have the following similarity variables for the equation (1)

$$\xi = \frac{x}{t^{\frac{1}{3}}} - \frac{3}{2}\frac{t^{\frac{2}{3}}}{\mu},$$

$$q(x,t) = \frac{3}{4}\frac{\left(\frac{x}{t^{\frac{1}{3}}} - \frac{3}{2}\frac{t^{\frac{2}{3}}}{\mu}\right)t^{\frac{1}{3}}}{a\mu} + \frac{9t}{16a\mu^2} + \frac{F(\xi)}{t^{\frac{1}{3}}}, \quad (55)$$

where ξ is new independent variable and F is new dependent variable.
Using (55) in Eq. (1), we have

$$3bF''' + 3aF'^2 - \xi F' - F = 0, \tag{56}$$

where $(')$ denotes derivative with respect to ξ.
To obtain a solutions of Eq. (56), we assume that Eq. (56) admits a solution of the form

$$F(\xi) = \frac{b_0}{\xi} + a_0 + a_1\xi + a_2\xi^2, \tag{57}$$

where b_0, a_0, a_1 and a_2 are to be determined. Substituting Eq. (57) into Eq. (56), we obtain the following two cases:

Case i: $b_0 = \frac{6b}{a},\ a_0 = a_1 = a_2 = 0$;

Case ii: $b_0 = a_0 = a_1 = 0,\ a_2 = \frac{1}{4a}$.

For cases (i) and (ii), the corresponding solutions of main Eq. (1) are given as

$$q(x,t) = \frac{3(8x^2\mu^2 - 18tx\mu + 9t^2 + 64b\mu^3)}{16\mu^2 a\left(2x\mu - 3t\right)},$$

$$q(x,t) = \frac{x^2}{4ta}. \tag{58}$$

5.1.2. Vector Field $V_5 + \alpha V_2 + \beta V_1$

Corresponding to this vector field, we have the following similarity variables for the equation (1)

$$\sigma = \beta x - \alpha t,$$

$$q\left(x,t\right) = \frac{x}{\alpha} + G\left(\sigma\right), \tag{59}$$

where σ is a new independent variable and $G(\sigma)$ is a new dependent variable. Using similarity variables (59) in equation (1), we have the following ODE

$$\beta^3 G''' + a\left(\frac{1}{\alpha} + \beta G'\right)^2 - \alpha G' = 0, \tag{60}$$

where $(')$ denotes the derivative with respect to σ.

Let Eq. (60) admits a solution of the form

$$G(\sigma) = a_0 + a_1\sigma, \tag{61}$$

where a_0 and a_1 are to be determined.

Substituting G from Eq. (61) into Eq. (60), we have

$$a_0 = \text{arbitrary}, \quad a_1 = \frac{\alpha^2 - 2a\beta \pm \sqrt{\alpha^4 - 4\alpha^2 a\beta}}{2a\beta^2\alpha}. \tag{62}$$

The corresponding solutions of main equation (1) are

$$q(x,t) = \frac{x}{\alpha} + a_0 + \frac{\alpha^2 - 2a\beta \pm \sqrt{\alpha^4 - 4\alpha^2 a\beta}}{2a\beta^2\alpha}(\beta x - \alpha t). \tag{63}$$

5.1.3. Vector Field $V_1 + \gamma V_2$

The corresponding similarity variables are

$$\rho = x - \gamma t$$

$$q(x,t) = J(\rho), \tag{64}$$

where ρ is a new independent variable and J is a new dependent variable.

Substituting similarity variables (64) into main equation (1), we have

$$-\gamma J' + aJ'^2 + bJ''' = 0, \tag{65}$$

where $(')$ denotes the derivative with respect to ρ.

The corresponding solutions of main equation (1) are

$$\text{(i)} \quad q(x,t) = C_2 + 3b\gamma \tanh\left(C_1 + \frac{\gamma(x-\gamma t)}{2\sqrt{b\gamma}}\right)\frac{1}{a\sqrt{b\gamma}}, \tag{66}$$

$$\text{(ii)} \quad q(x,t) = C_1 + \frac{\gamma}{a}(x - \gamma t), \tag{67}$$

where C_1 and C_2 are arbitrary constants.

6. NUMERICAL SIMULATION

This section is going to address the numerical simulation of the p-KdV equation that is given by (1) for $x_L \le x \le x_R$. An explicit numerical scheme will be devoped and a topological 1-soliton solution will be given. The initial condition is taken to be

$$q(x,0) = A\tanh(Bx). \tag{68}$$

The approximate solution will be designated by Q_m^n while the exact solution will be denoted by q_m^n. The proposed scheme will be displayed at the grid point (x_m, t_n). Therefore,

$$\frac{1}{k}\delta_t Q_m^n + \frac{a}{h^2}\left(\delta_x Q_m^n\right)^2 + \frac{b}{2h^3}\delta_{xxx}^3 Q_m^n = 0, \tag{69}$$

where

$$\delta_t Q_m^n = Q_m^{n+1} - Q_m^n, \tag{70}$$

$$\delta_x Q_m^n = Q_{m+1}^n - Q_m^n, \tag{71}$$

and

$$\delta_{xxx}^3 Q_m^n = Q_{m+2}^n - 2Q_{m+1}^n + 2Q_{m-1}^n - Q_{m-2}^n. \tag{72}$$

This is an explicit scheme that is of second order in space and time. By choosing the parameters $h = 0.1$, $k = 0.01$, $a = 1.0$, $b = 1.0$, $B = 0.5$, $x_L = -x_R = -50$ and $0 \le t \le 40$, Fig. 1 displays the numerical simulation for the topological soliton solution.

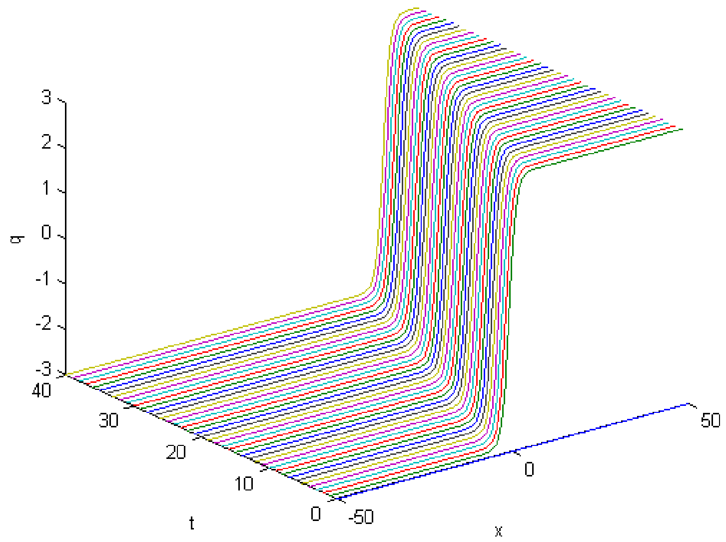

Fig. 1 – Numerical solution for the topological soliton *q*.

7. CONCLUSIONS

This paper addressed the potential Korteweg-de Vries equation by using several forms of integration architecture. They are the *ansatz method*, *mapping method* and the *Lie symmetry analysis*. These methods lead to several solutions for the potential Korteweg-de Vries equation. Such solutions include shock waves or kinks or topological soliton, cnoidal and snoidal waves as well as other rational solutions. To the best of our knowledge, these solutions were obtained for the first time in this paper. These solutions are going to be very useful in further investigation into the potential Korteweg-de Vries equation. In future there are several issues that can be addressed. Some of them are the study of conservation laws, the perturbation analysis, as well as potential Korteweg-de Vries equation with random coefficients. These open issues just form the tip of the iceberg.

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