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MEAN SQUARE SOLUTIONS OF SECOND-ORDER RANDOM DIFFERENTIAL EQUATIONS BY USING HOMOTOPY ANALYSIS METHOD

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Abstract. In this paper, the Homotopy Analysis Method (HAM) is successfully applied for solving second-order random differential equations, homogeneous or inhomogeneous. Expectation and variance of the approximate solutions are computed. Several numerical examples are presented to show the ability and efficiency of this method.

Key words: Random differential equations; Stochastic differential equation; Homotopy Analysis Method; Uniform random variable and Beta random variable.

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1. INTRODUCTION

A random ordinary differential equations are an ordinary differential equations which contains random constants or random variables. Most scientific problems, biology, engineering and physical phenomena occur in the from of random differential equations [1–3]. In refs [4–11], several first-order random differential models are solved using mean square calculus. HAM is a powerful mathematical technique and has been already applied to several nonlinear problems [12, 13]. Many scientific models is described as a nonlinear second order random differential equation. Recently, several second-order random differential models are solved using mean square random Variational Iteration Method Expectation and Variance of the approximate solutions has been computed [14–20]. As a pursue of this work, we have solved mean square for second-order random initial value problems using HAM. Comparison between the Variational Iteration Method and HAM shows that the HAM agrees well with the exact one.

The plan of our paper is as follows:

In section 2 we study HAM as methods of solving differential. In section 3 the statistical functions of the mean square random is solved using Homotopy Analysis Method. Finally in section 4 we adopt several examples to illustrate the using of HAM for approximating the mean and the variance.

2. HOMOTOPY ANALYSIS METHOD

The homotopy analysis method (HAM) has been proposed by Liao in [13]. The HAM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. Consider the following differential equation

$$N[X(t)] = 0, \tag{1}$$

where N is a nonlinear auxiliary operator and $X(t)$ is an unknown function. We ignore the boundary and initial conditions, which can be treated in the similar way. The zero-order deformation equation is constructed as

$$(1-q)L[x_n(t) - x_{n-1}(t)] = q\hbar N[x(t), A], \tag{2}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ a nonzero auxiliary parameter, L an auxiliary linear operator, $X(t)$ is an unknown function. It is important to have enough freedom to choose auxiliary unknown in HAM. By Taylor's theorem, $X(t)$ can be expanded with respect to the embedding parameter q as

$$X(t) = X_0(t) + \sum_{n=1}^{\infty} X_n(t) q^n \tag{3}$$

$$X_n(t) = \frac{1}{n!} \frac{\partial^n X(t)}{\partial q^n}|_{q=0} \tag{4}$$

Differentiating the Zeroth-Order deformation equation n-times with respect to q at $q = 0$ and then dividing it by $n!$, we have the following nth-order deformation equation

$$L[X_n(t) - X_{n-1}(t)] = \hbar R_n(X(t)), \tag{5}$$

where

$$R_n(X(t)) = \frac{1}{(n-1)!} \frac{\partial^n - 1N[X(t)]}{\partial q^n - 1}|_{q=0}. \tag{6}$$

If the series (3) converges at $q = 1$ we have

$$X(t) = X_0(t) + \sum_{n=1}^{\infty} X_n(t). \tag{7}$$

To illustrate the basic concept of the HAM. We consider the second order random differential equation with initial conditions in the following from

$$\frac{d^2 x_n(t)}{dt^2} = (1+h)\frac{d^2 x_{n-1}(t)}{dt^2} + hN[x(t), A] \tag{8}$$

$$x(0) = Y_0, \frac{dx(t)}{dt} |_{t=0} = Y_1, \tag{9}$$

where $L[x(t)] = \frac{d^2x(t)}{dt^2}$, $N[x(t), A]$ is a nonlinear operator and $x_0(t) = Y_0 + Y_1 t$.

3. STATISTICAL FUNCTIONS OF THE MEAN SQUARE RANDOM HAM

This section concern with the computation of the main statistical functions of the m.s solution of Eq. (1) given by the iteration Eq. (2).

$$\begin{aligned}
E[\frac{d^2x_n(t)}{dt^2}] &= (1+h)E[\frac{d^2x_{n-1}(t)}{dt^2}] + hE[N[x_n(t), A]], \\
E[\frac{d^2x_n(t)}{dt^2}]^2 &= [(1+h)E[\frac{d^2x_{n-1}(t)}{dt^2}]]^2 + [hE[N[x_n(t), A]]^2 \\
&\quad + 2(1+h)hE[([N[x_n(t), A])(\frac{d^2x_{n-1}(t)}{dt^2})], \\
V[\frac{d^2x_n(t)}{dt^2}] &= (1+h)V[\frac{d^2x_{n-1}(t)}{dt^2}] + V[hN[x_n(t), A]] \\
&\quad + 2\,\mathrm{Cov}[h(1+h)(\frac{d^2x_{n-1}(t)}{dt^2})(N[x_n(t), A])].
\end{aligned} \tag{10}$$

The following lemma guarantee the convergent of the sequence $E[X_n(t)]$ to $E[X(t)]$ and the sequence $V[X_n(t)]$ to $V[X(t)]$ if the sequence the $X_n(t)$ converges to $X(t)$.

3.1. LEMMA

Let X_n and Y_n be two sequences of $2-r, vsX$ and Y, respectively, *i.e.*: $\lim_{n\to\infty} X_n = X$ and $\lim_{n\to\infty} Y_n = Y$ then $\lim_{n\to\infty} E[X_n Y_n] = E[XY]$. If $X_n = Y_n$, then $\lim_{n\to\infty} E[X_n^2] = E[X^2]$, $\lim_{n\to\infty} E[X_n] = E[X]$ and $\lim_{n\to\infty} V[X_n] = V[X]$.

4. ILLUSTRATIVE EXAMPLES

In this section, we adopt several examples to illustrate the using of homotopy analysis method for approximating the mean and the variance using `Mathematica`.

4.1. EXAMPLE 1

Consider random initial value problem

$$\frac{d^2X(t)}{dt^2} + A^2X(t) = 0, \quad X(0) = Y_0 \text{ and } \frac{dx(t)}{dt} |_{t=0} = Y_1,$$

where A^2 is the Beta random variable with ($\alpha = 2$, $\beta = 1$) and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$,

$E[Y_1^2] = 3$ and $E[Y_0Y_1] = 0$.
We have presented the results in Figs. (1) & (2).

$$E[x[t]] = 1+t-\frac{2}{27}t^2(3+t)+\frac{2t^4(5+t)}{1215}-\frac{4t^6(7+t)}{229635}+\frac{2t^8(9+t)}{18600435}+\dots \qquad (11)$$

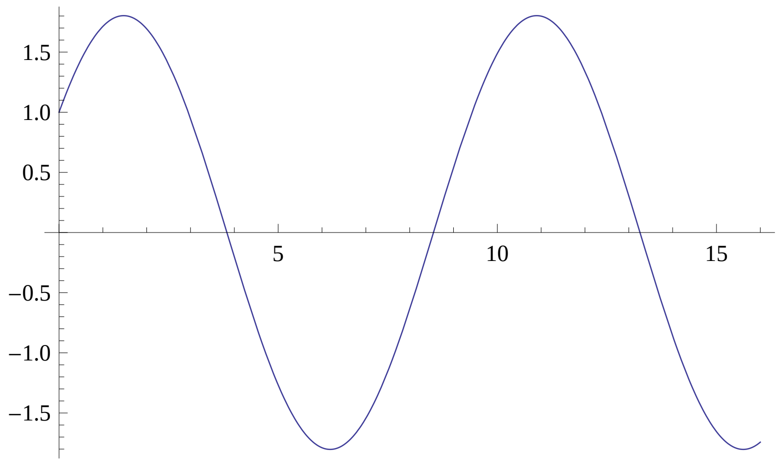

Fig. 1 – Graphs of the expectation approximation solution from the HAM with $n = 18$.

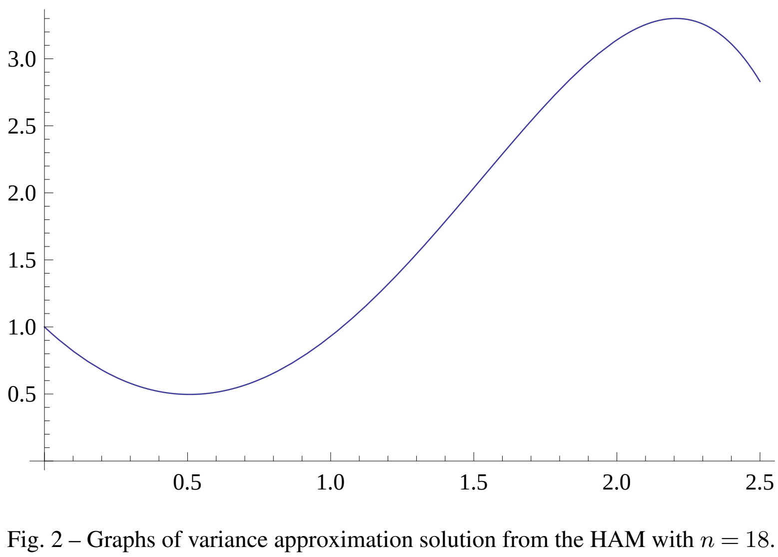

Fig. 2 – Graphs of variance approximation solution from the HAM with $n = 18$.

4.2. EXAMPLE 2

Consider random initial value problem

$$\frac{d^2X(t)}{dt^2} + AtX(t) = 0, \quad X(0) = Y_0 \text{ and } \frac{dx(t)}{dt}\,|_{t=0} = Y_1,$$

where A is a Beta r.v. with parameters $\alpha = 2$ and $\beta = 3$ and the initial conditions Y_0 and Y_1 are independent r.v.'s such as $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 2$, $E[Y_1^2] = 5$. We have presented the result in Figs. (3) and (4).

$$E[x[t]] = 1 + 2t - \frac{t^3(1+t)}{15} - \frac{t^9(7+4t)}{1417500} + \frac{t^6(7+5t)}{7875} + \dots \tag{12}$$

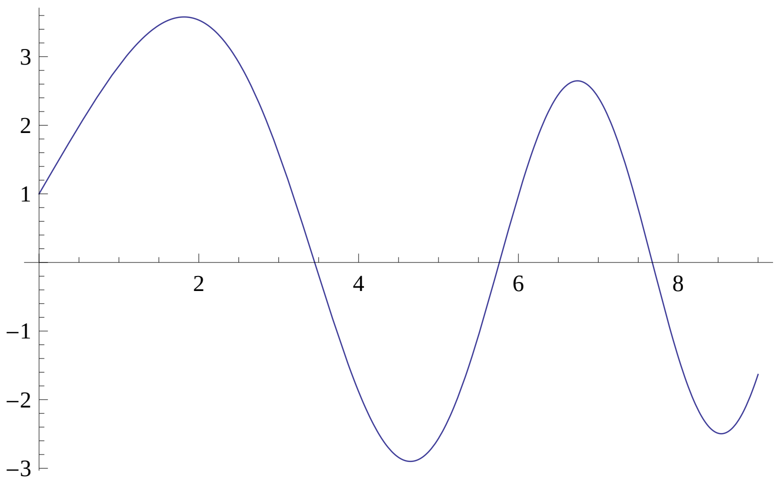

Fig. 3 – Graphs of the expectation approximation solution from the HAM with $n = 18$.

4.3. EXAMPLE 3

Consider the problem

$$\frac{d^2X(t)}{dt^2} + 2A\frac{dX(t)}{dt} + A^2X(t) = 0, \; X(0) = Y_0 \text{ and } \frac{dx(t)}{dt}\,|_{t=0} = Y_1,$$

where A is a Beta r.v. with parameters $\alpha = 2$ and $\beta = 1$ and independently of the initial conditions Y_0 and Y_1, which are independent r.v.'s, satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 1$.

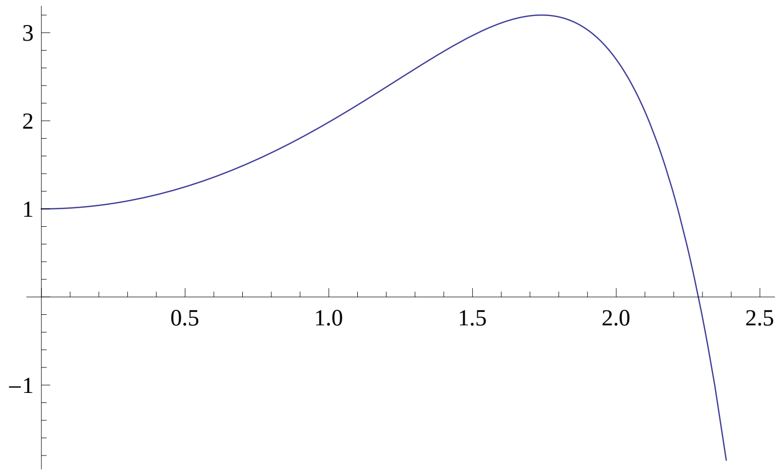

Fig. 4 – Graphs of variance approximation solution from the HAM with $n = 18$.

We have presented the result in Figs. (5) and (6)

$$E[x[t]] = 1 + t - \frac{2}{27}t^2(12+t) + \frac{2t^3(240+35t+t^2)}{1215} - \frac{4t^4(7560+1386t+70t^2+t^3)}{229635} + \ldots \quad (13)$$

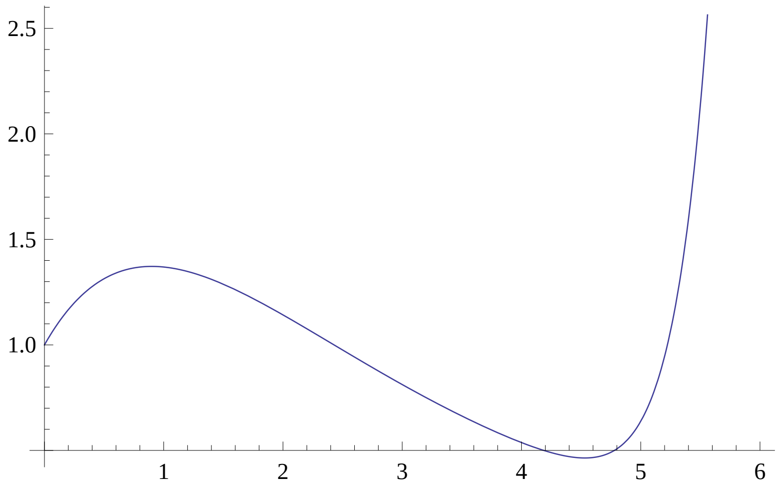

Fig. 5 – Graphs of the expectation approximation solution from the HAM with $n = 16$.

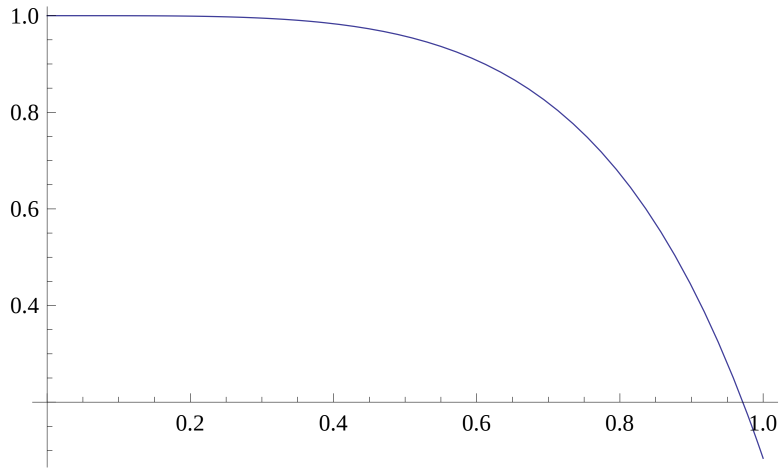

Fig. 6 – Graphs of variance approximation solution from the HAM with $n = 16$.

4.4. EXAMPLE 4

Consider the problem

$$\frac{d^2X(t)}{dt^2} + At\frac{dX(t)}{dt} = 0,\ X(0) = Y_0 \text{ and } \frac{dx(t)}{dt}\ |_{t=0}= Y_1,$$

where A is a Uniform r.v. with parameters $\alpha = 0$ and $\beta = 1$, and independently of the initial conditions Y_0 and Y_1, which are independent r.v.'s, satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 1$.
We have presented the result in Figs. (7) and (8).

$$E[x[t]] = 1 + t - \frac{t^3}{12} + \frac{t^5}{160} - \frac{t^7}{2688} + \frac{t^9}{27648} - \frac{t^{11}}{1351680} + \frac{t^{13}}{38338560} + ... \quad (14)$$

4.5. EXAMPLE 5

Consider the following problem

$$\frac{d^2X(t)}{dt^2} + AX(t) = 0,\ \ X(0) = Y_0\ \ and\ \ \frac{dx(t)}{dt}\ |_{t=0}= Y_1,$$

where A is a Uniform r.v. with parameters $\alpha = 0$ and $\beta = 2$, and independently of the initial conditions Y_0 and Y_1, which are independent r.v.'s, satisfy $E[Y_0] = 1$, $E[Y_0^2] = 4$, $E[Y_1] = 1$, $E[Y_1^2] = 2$.

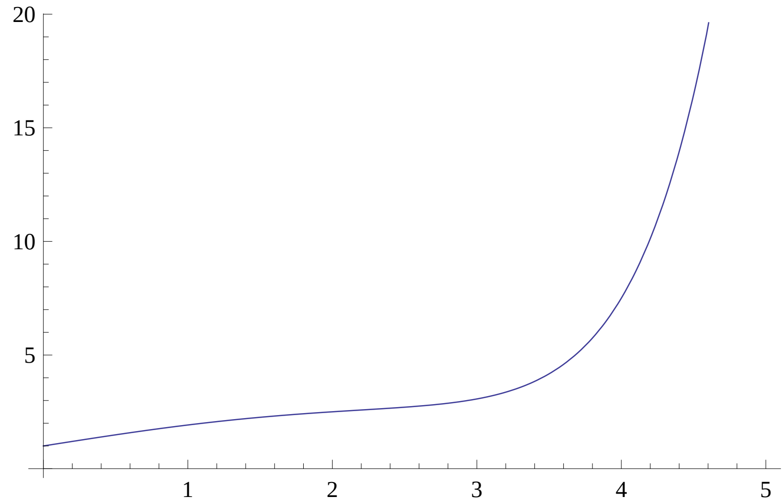

Fig. 7 – Graphs of the expectation approximation solution from the HAM with $n = 20$.

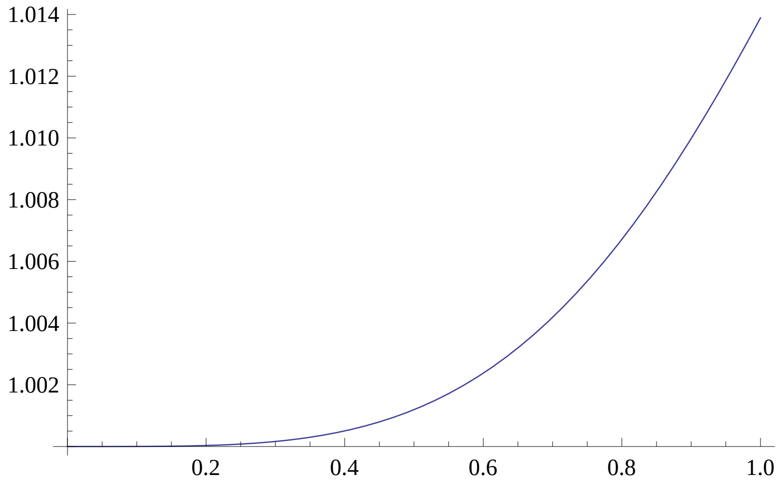

Fig. 8 – Graphs of variance approximation solution from the HAM with $n = 20$.

We have presented the result in Figs. (9) and (10).

$$\begin{aligned} E[x[t]] = 1 + t - \frac{t^2}{6}(3+t) + \frac{t^4}{120}(5+t) - \frac{t^6(7+t)}{5040} + \frac{t^8(9+t)}{362880} \\ - \frac{t^{10}(11+t)}{39916800} + \frac{t^{12}(13+t)}{6227020800} - \frac{t^{14}(15+t)}{1307674368000} \\ + \frac{t^{16}(17+t)}{355687428096000} - \frac{t^{18}(19+t)}{121645100408832000} + \ldots \end{aligned} \tag{15}$$

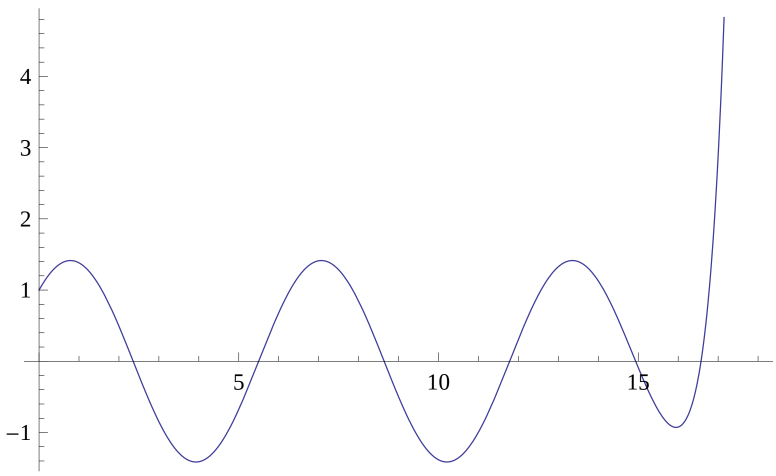

Fig. 9 – Graphs of the expectation approximation solution from the HAM with $n = 20$.

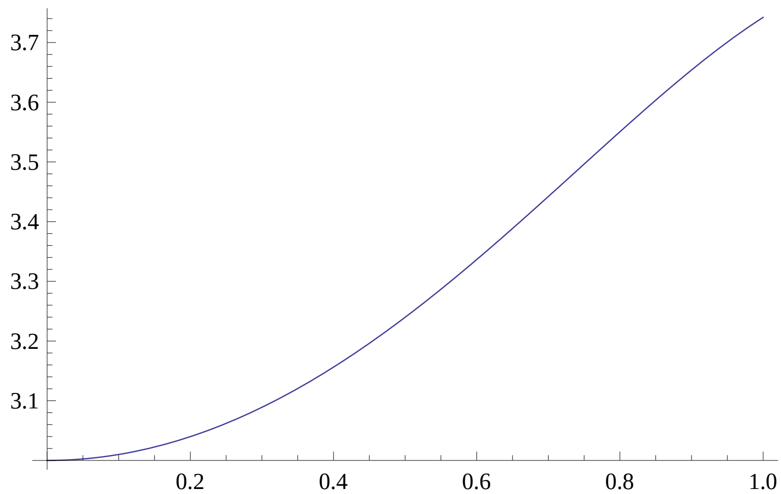

Fig. 10 – Graphs of variance approximation solution from the HAM with $n = 18$.

4.6. EXAMPLE 6

Consider the problem

$$\frac{d^2X(t)}{dt^2} + AX(t) = -X(t) + \sin(t),\ X(0) = Y_0 \text{ and } \frac{dx(t)}{dt}\ |_{t=0}= Y_1,$$

where A is a Uniform r.v. with parameters ($\alpha = 1$ and $\beta = 2$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 6$ and $E[Y_0Y_1] = 0$.
We have presented the result in Figs. (11) and (12).

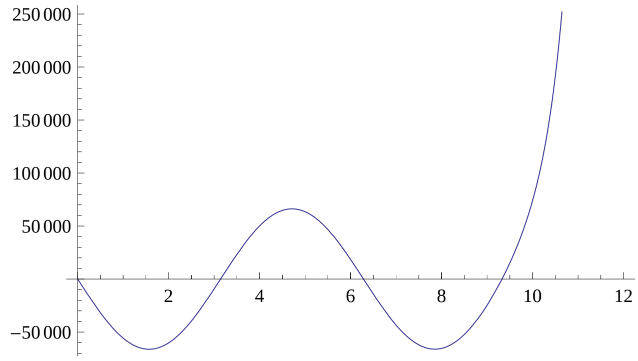

Fig. 11 – Graphs of the expectation approximation solution from the HAM with $n = 12$.

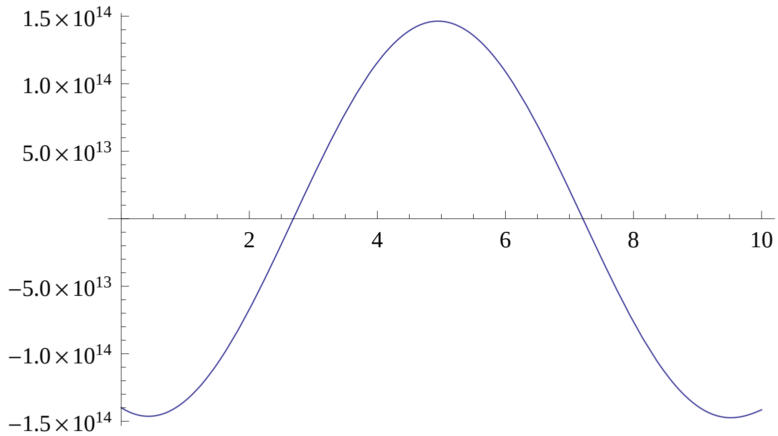

Fig. 12 – Graphs of variance approximation solution from the HAM with $n = 10$.

$$
\begin{aligned}
E[x[t]] = {} & 1+t-\frac{5}{12}(3+t)t^2+ \\
& + \frac{15625t^{24}(25+t)-16157239383760070469467347353 6\sin[t]}{40661706455989579897896 96} \\
& - \frac{78125t^{22}(23+t)+1346402730557959214472364032\sin[t]}{8471188844997829145395 2}+ \\
& + \frac{15625t^{20}(21+t)-532141509355537247502336\sin[t]}{837073996541287464 96} \\
& - \frac{15625t^{18}(19+t)+12668042903711502827 52\sin[t]}{498258331274575872} \\
& + \frac{3125t^{16}(17+t)-740529843298172928\sin[t]}{728447852740608}+\ldots
\end{aligned} \tag{16}
$$

5. CONCLUSION

In dealing with problems arises from real-world applications, it is only rarely possible to find the solution of a given random differential equation in closed form. Even when such an analytic solution is available,it is typically complicated to use in practice. As a result it is indispensable to have a number of numerical algorithms so that one is able to calculate numerical solutions with sufficient accuracy in a reasonable time. In some cases an approximate solution may be more useful than an numerical solution. Having the above considerations in mind, in this study, we have applied the iterative method HAM for finding analytical approximation solutions to the stochastic differential equation. We have calculated the expectation and variance approximate solutions with `Mathematica`.

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