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CONSTRUCTION OF SOLITON SOLUTION TO THE KADOMTSEV-PETVIASHVILI-II EQUATION USING HOMOTOPY ANALYSIS METHOD

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Abstract. The Kadomtsev-Petviashvili (KP) equation is one of the most universal models in nonlinear wave theory, which arises as a reduction of system with quadratic nonlinearity which admit weakly dispersive waves, in a paraxial wave approximation. In this paper the homotopy analysis method (HAM) is applied to obtain approximate solution of KP-II equation. The series solution is developed and the recurrence relations are given explicitly. The results obtained ensure that this method is capable for solving a larger number of nonlinear partial differential equation that have with the application in physics and engineering. Numerical solution obtained by HAM is compared with exact solution.

Key words: homotopy analysis method, Kadomtsev-Petviashvili-II equation, soliton solution, h-curve.

1. INTRODUCTION

Most scientific problems and physical phenomena occur nonlinearly. In the past two decades, the discovery of soliton solutions for certain (1+1)-dimensional nonlinear evolution equations with physical applications has aroused great interest and attention among physists and mathematicians [31]. Kadomtsev-Petviashvili (KP) equations are universal models for dispersive, weakly nonlinear waves, which are essentially One-dimensional, when weak transverse effect are taken into account [19, 20]. The KP equation originates from a 1970 paper by two Soviet physicists, Boris Kadomtsev and Vladimir Petviashvili. The two researchers derived the equation that now bears their name as a model to investigate the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. In the absence of transverse dynamics, this problem is explained by the Korteweg-de Vries (KdV) equation. The KP equation was soon widely accepted as a natural extension of the classical KdV equation to two spatial dimensions, and was later derived as a model for surface and internal water waves by Ablowitz and Segur [2], and in nonlinear optics by Pelinovsky et al. [34], as well as in other physical settings. The KP equation is also used as a classical model for developing and testing new mathematical techniques, *e.g.* in problems of well-possedness in non-classical function spaces [38], in applications of the dynamical system methods for water waves [14], and in the variational theory of existence and stability of energy minimizers [11]. This equation is a nonlinear partial differential equation in two spatial and one temporal coordinate which describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate. There are two distinct versions of the KP equation, which can be written in normalized form as follows

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{1}$$

Here x and y are respectively the longitudinal and transverse spatial coordinates, subscripts x, y and t denote partial derivatives, and $\sigma^2 = 1$. The case $\sigma = 1$ is known as the KP-II equation, and models, for instance, water waves with small surface tension. The case $\sigma = i$ is known as the KP-I equation, and may be used to model waves in thin films with high surface tension [7]. Here we consider the KP-II equation. The KP-II equation is a universal integrable system in two spatial dimensions in the same way that the KdV equation can be regarded as a universal integrable system in one spatial dimension, since many other integrable systems can be obtained as reductions. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations (PDEs) in physics and mathematics is still a big problem that needs new methods to discover new exact or approximate solutions. There are also some analytic techniques for nonlinear equations. A variety of effective analytical and semi-analytical methods have been developed to be used for solving nonlinear PDEs, such as the homotopy perturbation method (HPM) [15, 23–30], the variational iteration method (VIM) [17], the sine-cosine method [5], Adomian decomposition method (ADM) [4, 35, 37] and others [10, 18, 36]. Recently, homotopy analysis method (HAM) has been successfully employed to solve many types of nonlinear problems in science and engineering [1,9,12,16]. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h-curve, it is easy to determine the valid regions of h to gain a convergent series solution. This method has many advantages over the classical methods, mainly, it is independent of any small or large quantities.

The paper has been organized as follows. In section 2, The basic ideas of the present approach is described. In section 3, by choosing special forms of initial conditions, the proposed method is applied to study KP-II equation. In section 4, the convergence of the HAM series solution is analyzed. The paper is concluded in section 5.

2. BASIC IDEA OF HAM

We consider the following differential equation

$$N[u(x,t)] = 0, \tag{2}$$

where N is a nonlinear operator, t is independent variable, and $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can treated in the similar way. By means of generalizing the traditional homotopy method, Liao [22] constructs the so-called zero-th order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)], \tag{3}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(x,t) \neq 0$ is a nonzero auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$, and $\phi(x,t;q)$ is a unknown function. It is important that ones has great freedom to choose auxiliary parameters in HAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t), \tag{4}$$

Thus, as q increases from 0 to 1, The solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding by Taylor series with respect to q, we have

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, q^m, \tag{5}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \mid_{q=0}. \tag{6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (5) converges at $q = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{7}$$

which must be one of solution for the original nonlinear equation, as proved by Liao [22]. As $h = -1$ and $H(x,t) = 1$, equation (3) becomes

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] + qN[\phi(x,t;q)] = 0, \tag{8}$$

which is used mostly in the homotopy perturbation method (HPM). According to the definition (6), the governing equation can be deduced from the zero-th order deformation equation (3). Define the vector

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}.$$

Differentiating equation (3) m times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, we have the so-called m^{th} order deformation equation

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = hH(x,t)R_m(\vec{u}_{m-1}), \tag{9}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{10}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ \\ 1 & m > 1. \end{cases} \tag{11}$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (9) under the linear boundary condition that come from original problem, which can be easily symbolically solved by MATLAB computer software.

3. APPLICATION

In this section we employ our algorithm of the homotopy analysis method to find out series solutions of the KP-II equation, as the following

$$u_{xt} + 6u_x^2 + 6uu_x + u_{xxxx} + 3u_{yy} = 0, \tag{12}$$

subject to initial condition

$$u(x,y,0) = 2/\cosh^2(x), \tag{13}$$

where λ is an arbitrary constant.
To solve the equation (12) by means of HAM, according to the initial condition denoted in equation (13), it is natural to choose

$$u_0(x,y,t) = 2/\cosh^2(x),$$

we choose the linear operators

$$L[\phi(x,y,t;q)] = \frac{\partial^2\phi(x,y,t;q)}{\partial x \partial t}.$$

From (12), we define nonlinear operators

$$N[\phi] = \phi_{xt} + 6\phi_x^2 + 6\phi\phi_{xx} + \phi_{xxxx} + 3\phi_{yy}.$$

We construct the zero-order deformation equation

$$(1-q)L[\phi(x,y,t;q) - u_0(x,y,t)] = qhN[\phi], \tag{14}$$

Differentiating (14), m times with respect to q, then setting $q=0$ and finally dividing them by $m!$, we gain the m^{th} order deformation equation

$$L[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = hR_m(\vec{u}_{m-1}), \tag{15}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial x \partial t} + 6\sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x}\frac{\partial u_{m-1-i}}{\partial x} + 6\sum_{i=0}^{m-1} u_i \frac{\partial^2 u_{m-1-i}}{\partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} + 3\frac{\partial^2 u_{m-1}}{\partial y^2}$$

Now the solution of the m^{th} order deformation equation (15) as the following form

$$\begin{aligned} u_m(x,y,t) &= \chi_m u_{m-1}(x,y,t) + hL^{-1}[R_m(\vec{u}_{m-1})], \\ u_0(x,y,t) &= 2/\cosh^2(x), \\ u_1(x,y,t) &= -16ht\sinh(x)/\cosh^3(x), \\ u_2(x,y,t) &= \frac{-1}{\cosh^4 x}\left[8ht(\sinh(2x) + h\sinh(2x) + 4ht) - 64h^2t^2(\cosh^2 x - 1)\right], \\ &\vdots \end{aligned}$$

Therefore, using equation $u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots$, we have

$$u(x,y,t) = \frac{64h^2t^2+2}{\cosh^2 x} - \frac{8ht(2\sinh(2x) + h\sinh(2x) + 12ht)}{\cosh^4 x} + \ldots.$$

Table 1.

Absolute errors for $h=-0.8$ when $x=20$ and $y=20$.

t	Absolute error
0.02	3.9837e-019
0.08	5.1319e-018
0.1	8.5898e-018
0.2	5.4307e-017
0.5	2.4470e-016

4. CONVERGENCE OF THE ANALYTIC SOLUTION

The solutions given by the HAM contain an auxiliary parameter h, which can be used to control and adjust the convergence region and rate of the HAM solution series. It is interesting that the convergence rate of the approximation series depends upon the value auxiliary parameter h. In general, by means of the so-called h-curve, it is straightforward to choose an appropriate range for h which ensures the convergence of the solution series. To study the influence of h on the convergence of

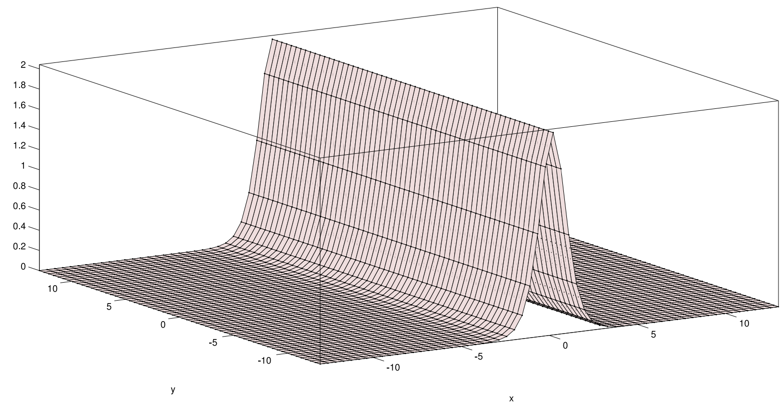

Fig. 1 – The 3$^{\rm rd}$ order HAM solution of $u(x, y, t)$ when $t = 0.05$ and $h = -1$.

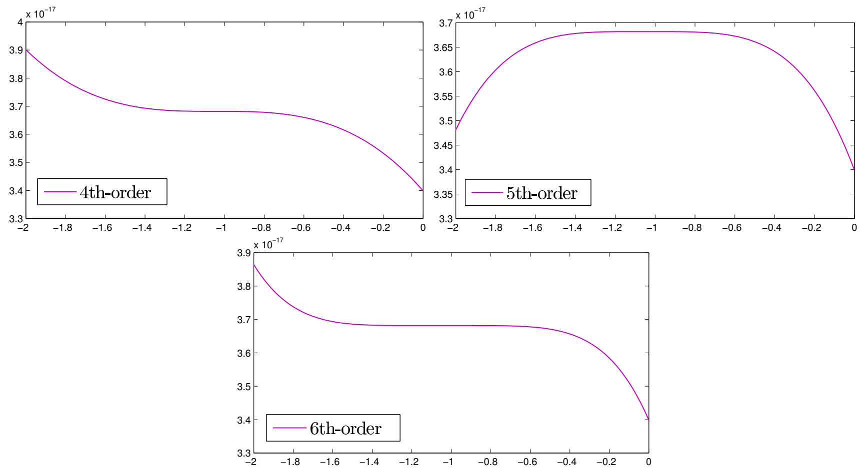

Fig. 2 – The h-curve obtained from the HAM of equation (12) in the case $x = 20$, $y = 20$ and $t = 0.01$.

solution, the h-curves of the 4$^{\rm th}$ order, 5$^{\rm th}$ order and 6$^{\rm th}$ order approximate solutions are plotted for $x = 20$, $y = 20$ and $t = 0.01$, as shown in Fig. 2. It is easy to see that in order to have a good approximation, h has to be chosen in $[-1.12, -0.85]$. Thus, the auxiliary parameter h plays an important role within the frame of the HAM. The absolute error for differences between the exact solution [6] and the 3$^{\rm rd}$ order approximate solution given by HAM are represented in Table 1.

5. CONCLUSION

The applications of the homotopy analysis method (HAM) were extended successfully for solving Kadomtsev-Petviashvili-II equation. The HAM provides us with a convenient way to control the convergence of approximation series by adapting h, which is a fundamental qualitative difference in analysis between HAM and other methods. The work emphasized our belief that the method is a valid technique to handle nonlinear partial differential equations. The basic ideas of this approach can be widely employed to solve other strongly nonlinear problems. Moreover, results indicate that the solution obtained by this method converges rapidly to an exact solution and Table 1 confirms it.

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