Dedicated to Professor Ioan-Iovitz Popescu's 80th Anniversary

THE REFLECTIVITY AND TRANSMISSIVITY - QUANTUM VIEW

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Abstract. The electromagnetic energy transfer between two dielectrics is analyzed using quantum theory. Our approach does not use field theories and Maxwell's equations, but only the methods of quantum mechanics.

Key words: d'Alembert type equation, Fresnel formulas, step potential reflectivity, transmissivity.

1. INTRODUCTION

Classical optics allows the assessment of reflection and transmission coefficients at the boundary of two dielectrics for parallel and perpendicular polarizations. These equations are known as Fresnel's relations; they express the transfer of electromagnetic energy at the dielectric interface. Next we summarize the reasoning for obtaining classical reflectivity and transmissivity. Detailed calculations can be found for example in [1].

The intensity of a light beam is the energy carried by the wave in unit time through unit area of cross section. It is proportional to the square amplitude of the electric field. If we deal with a harmonic plane wave, the Poynting vector $\vec{S}$ is written

$$\vec{S} = \frac{n}{2c}|E_0|^2\,\vec{k},\tag{1}$$

where n, c, E_0 and $\vec{k}$ are respectively the refractive index of the environment, the light speed in vacuum, the electric field amplitude and the wave vector.

Power transported through the area element $d\vec{A}$, perpendicular to the surface is

$$P = \vec{S} \cdot d\vec{A}\tag{2}$$

or

$$P = \frac{n}{2c}|E_0|^2 \vec{k} \cdot d\vec{A}$$ (3)

and if we denote by θ the angle between the direction of wave propagation given by vector $\vec{k}$ and vector $d\vec{A}$

$$P = \frac{n}{2c}|E_0|^2 \cos\theta \, dA.$$ (4)

The incident P_{inc}, reflected P_{refl}, and transmitted P_{trans} power at dielectric interface, are respectively

$$P_{inc} = \frac{n_1}{2c}|E_{i0}|^2 \cos\theta_1 \, dA, \, P_{refl} = \frac{n_1}{2n_0}|E_{r0}|^2 \cos\theta_1 \, dA \text{ and } F_{trans} = \frac{n_2}{2n_0}|E_{t0}|^2 \cos\theta_2 \, dA$$

With these specifications reflectivity becomes

$$\mathcal{R} = \frac{P_{refl}}{P_{inc}} = \frac{|E_{r0}|^2}{|E_{i0}|^2} = |R|^2$$ (5)

whereas transmissivity is given by

$$\mathcal{T} = \frac{P_{trans}}{P_{inc}} = \frac{n_2 \cos\theta_2 |E_{t0}|^2}{n_1 \cos\theta_1 |E_{i0}|^2} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1}|T|^2$$ (6)

In the following we will try to find these relationships exclusively by quantum mechanics methods.

2. THE PHOTON IN A DIELECTRIC ENVIRONMENT

The equation that describes the motion of a photon in a certain dielectric are found in [2]. Photon wave equationis

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi + \frac{1}{c^2}(1 - n^2)\frac{\partial^2 \psi}{\partial t^2} = 0,$$ (7)

where *n* is the refraction index of the dielectric. If we take into consideration that.

$$\frac{\partial}{\partial t} = -\frac{i}{\hbar}\bar{E} \rightarrow -i\omega$$ (8)

with $\bar{E}$ energy operator and with the notation

$$K \equiv \frac{\omega}{c}\sqrt{1 - n^2}$$ (9)

wave equation becomes

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi - K^2\psi = 0. \tag{10}$$

We would like to point out that this D'Alembert type equation is relativistically correct unlike the Schrödinger equation which comes from classic quantum mechanics.

As the refractive index is smaller or bigger than unity, the choice of the sign in front of the last term in Eq. (10) makes the static solution of the Seelinger-Wed-Proca-Yukawa type respectively of the Helmholtz type.

In addition we consider that the photon moves in an environment of potential describing the dielectric environment [2]

$$U^2 = \hbar^2\omega^2(1 - n^2). \tag{11}$$

For photons probability current density we use the expression

$$\vec{j} = 2\omega^2 cn|\psi_0|^2. \tag{12}$$

deduced in the Annex.

3. TRANSMISSIVITY AND REFLECTIVITY AT THE INTERFACE OF TWO DIELECTRICS

Dielectric environment acts relative to photons as a potential of form (11). Therefore a set of two dielectrics can be viewed as a step potential
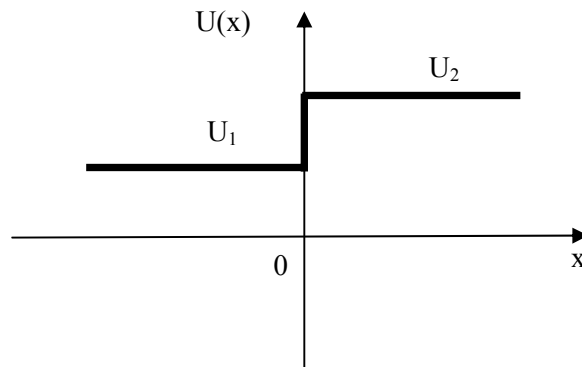

Fig. 1 – Step potential associated to the interface of two dielectrics $U_{1,2} = \hbar\omega\sqrt{1 - n_{1,2}^2}$.

At normal incidence on the separation surface between two dielectrics the transmissivity and reflectivity are respectively [3]

$$\mathcal{R} = \frac{j_{refl}}{j_{inc}} \tag{13}$$

and

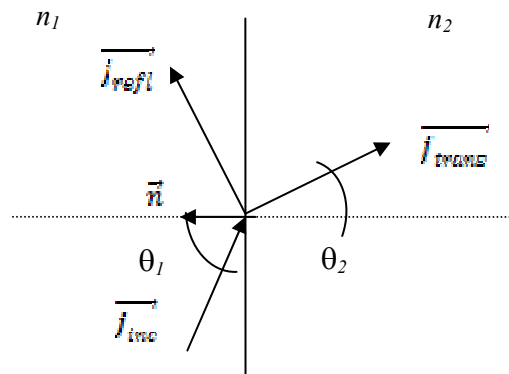$$\mathcal{T} = \frac{j_{trans}}{j_{inc}}. \tag{14}$$

Fig.2 Probability current density for incident, reflected and transmitted photons; $\vec{n}$ is the normal vector to the separation surface between dielectrics.

In the case of Fig. 2 when current densities vectors are not normal to the separation surface relations (13) and (14) are rewritten as:

$$\mathcal{R} = \frac{\overrightarrow{j_{refl}} \cdot \vec{n}}{\overrightarrow{j_{inc}} \cdot \vec{n}} \tag{15}$$

and

$$\mathcal{T} = \frac{\overrightarrow{j_{trans}} \cdot \vec{n}}{\overrightarrow{j_{inc}} \cdot \vec{n}}. \tag{16}$$

As the reflection angle is equal to the incidence angle θ_1, reflectivity is reduced to

$$\mathcal{R} = \frac{j_{refl}}{j_{inc}}. \tag{17}$$

On the other hand transmissivity is given by

$$\mathcal{T} = \frac{j_{trans}}{j_{inc}} \frac{\cos\theta_2}{\cos\theta_1}. \tag{18}$$

Using the current density expression (12) and Fresnel's relations [4]

$$D_\perp = -A_\perp \frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} \quad , C_\perp = A_\perp \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)}, \tag{19}$$

$$B_\parallel = A_\parallel \frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)} \quad \text{and} \quad C_\parallel = A_\parallel \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}, \tag{20}$$

where the constants *A*, *B* and *C* represent the magnitude of the incident, reflected, and transmitted wave function respectively. We can write the reflectivity and transmissivity for the cases of perpendicular and parallel polarization. In our specific case, $|\psi_0|^2$ will be replaced by the square moduli of the wave function amplitudes

$$\left|\psi_{0,\perp,\parallel}\right|^2_{inc} = \left|A_{\perp,\parallel}\right|^2, \quad \left|\psi_{0,\perp,\parallel}\right|^2_{ref} = \left|B_{\perp,\parallel}\right|^2 \left|\psi_{0,\perp,\parallel}\right|^2_{trans} = \left|C_{\perp,\parallel}\right|^2 \tag{21}$$

Next we rereplace from (15-18), expressions (12), and (17-20) with the notations from [4]

$$\left.\begin{array}{l}
\mathcal{R}_\perp = \dfrac{|B_\perp|^2}{|A_\perp|^2} = \left[\dfrac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}\right]^2 \\[3mm]
\mathcal{T}_\perp = \dfrac{j_{trans}\cos\theta_2}{j_{inc}\cos\theta_1}\dfrac{|C_\perp|^2}{|A_\perp|^2} = \dfrac{n_2\cos\theta_2}{n_1\cos\theta_1}\left[\dfrac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)}\right]^2 \\[3mm]
\mathcal{R}_\parallel = \dfrac{|B_\parallel|^2}{|A_\parallel|^2} = \left[\dfrac{\sin(\theta_1-\theta_2)\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}\right]^2 \\[3mm]
\mathcal{T}_\parallel = \dfrac{j_{trans}\cos\theta_2}{j_{inc}\cos\theta_1}\dfrac{|C_\parallel|^2}{|A_\parallel|^2} = \dfrac{n_2\cos\theta_2}{n_1\cos\theta_1}\left[\dfrac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}\right]^2
\end{array}\right\} \tag{22}$$

If we take into account the law of refractive this can also be deduced by quantum methodes [5]

$$\frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2}, \tag{23}$$

we finally get

$$\left.\begin{array}{l}
\mathcal{R}_\perp = \dfrac{\sin^2(\theta_1-\theta_2)}{\sin^2(\theta_1+\theta_2)} \\[3mm]
\mathcal{T}_\perp = \dfrac{\sin2\theta_1 \sin2\theta_2}{\sin^2(\theta_1+\theta_2)} \\[3mm]
\mathcal{R}_\parallel = \dfrac{tg^2(\theta_1-\theta_2)}{tg^2(\theta_1+\theta_2)} \\[3mm]
\mathcal{T}_\parallel = \dfrac{\sin2\theta_1 \sin2\theta_2}{\sin^2(\theta_1+\theta_2)\cos^2(\theta_1-\theta_2)}
\end{array}\right\} \tag{24}$$

Note identity of expressions with those obtained from electromagnetism [1].

4. COMMENTS AND CONCLUSIONS

Interaction between light and the propagation medium has found a complete settlement in classical electrodynamics, *via* Maxwell's equations. Photons as quantum particles, are massless and fieldless, and must obey the same equations of optics as light. As shown for example in [2], [4] and [5] this is possible. This last work, show once again that the photon feels a refractive index of the medium just as a quantum particle feels the crossed environment potential. Should not surprise those us. It is expected because relativistic quantum mechanics equations give the correct answer for two reasons: light is par excellence a relativistic phenomenon and electromagnetic waves consist of ensembles of photons.

ANNEX – Probability Current Density for Photons

In order to obtain the equation (12) let us consider Fig.A1. Let D be a domain bounded by surface Σ. We have a particle with wave function $\psi(\vec{x}, t)$, for which we want to know the probability of it being in the domain D

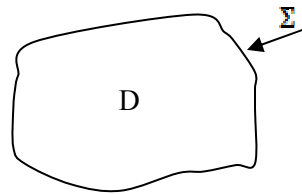$$Prob(D) = \int_D |\psi|^2 \, dr. \tag{A1}$$

Fig. 1A – Domain D bounded by surface Σ contains a variable number of particles in time.

As a particle can enter or exit from D $Prob(D)$ may vary over time. Since

$$|\psi|^2 = \psi^* \psi \tag{A2}$$

we can write

$$\frac{d}{dt}[Prob(D)] = \int_D \left(\frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t} \right) dr. \tag{A3}$$

We use equation (10) describing the photon displacement through a dielectric [2] where for $n > 1$

$$K \equiv i \frac{\omega}{c} \sqrt{n^2 - 1}. \tag{A4}$$

Some calculations lead us to

$$\Delta \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = K^2 \psi \ \text{ or } \ \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi - K^2 \psi. \tag{A5}$$

If we integrate by time we get

$$\int \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} dt = \int \nabla^2 \psi \, dt - \int K^2 \psi \, dt, \tag{A6}$$

$$\frac{\partial \psi}{\partial t} - c^2[\nabla^2 \int \psi \, dt - K^2 \int \psi \, dt], \tag{A7}$$

and after rearranging terms

$$\frac{\partial \psi}{\partial t} = c^2(\nabla^2 - k^2)(\int \psi \, dt), \tag{A8}$$

$$\frac{\partial \psi^*}{\partial t} = c^2(\nabla^2 - k^2)(\int \psi^* \, dt), \tag{A9}$$

$$\frac{d}{dt}[Prob(D)] = \int_R \{[c^2(\nabla^2 - k^2)(\int \psi^* \, dt)]\psi + \psi^*[c^2(\nabla^2 - k^2)(\int \psi \, dt)]\} dr, \tag{A10}$$

$$\frac{d}{dt}[Prob(D)] = c^2 \int_D \{\psi[\nabla^2(\int \psi^* \, dt)] + \psi^*[\nabla^2(\int \psi \, dt)]\} d^3x -$$

$$-c^2k^2 \int_D \{\psi[(\int \psi^* \, dt)] + \psi^*[(\int \psi \, dt)]\} d^3x. \tag{A11}$$

We also integrate by time the two equations $\frac{\partial \psi}{\partial t} = -i\omega\psi$ and $\frac{\partial \psi^*}{\partial t} = i\omega\psi^*$ obtain

$$\int \frac{\partial \psi}{\partial t} dt = -i\omega \int \psi dt, \tag{A12}$$

$$\int \frac{\partial \psi^*}{\partial t} dt = i\omega \int \psi^* dt, \tag{A13}$$

hence

$$\int \psi dt = i\omega\psi \text{ and } \int \psi^* dt = -i\omega\psi^*. \tag{A14}$$

In these conditions the second integral from $\frac{d}{dt}[Prob(D)]$ is canceled and results

$$\frac{d}{dt}[Prob(D)] = c^2 \int_D (-i\omega\psi\nabla^2\psi^* + i\omega\psi^*\nabla^2\psi)dr, \tag{A15}$$

$$\frac{d}{dt}[Prob(D)] = -i\omega c^2 \int_D (\psi\nabla^2\psi^* - \psi^*\nabla^2\psi)dr. \tag{A16}$$

Let's introduce $\vec{j}$ the probability of current density

$$\frac{d}{dt}[Prob(D)] = -\int_D \nabla \cdot \vec{j} \, dr. \tag{A17}$$

By identification, the current density becomes

$$\vec{j} = i\omega c^2 \left(\psi \nabla \psi^* - \psi^* \nabla \psi \right).$$ (A18)

Equations (8) (10) and (11) allow us to write the photon wave equation as

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi - \frac{1}{\hbar^2 c^2} U^2 \psi = 0,$$ (A19)

$$\Delta \psi - \frac{1}{\hbar^2 c^2} (E^2 - U^2) \psi = 0.$$ (A20)

We get another form using $E = \hbar\omega$

$$\Delta \psi - \frac{\omega^2}{c^2} n^2 \psi = 0,$$ (A21)

with the solution of the form of harmonic plane waves

$$\psi(\vec{r}, t) = \psi_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$ (A22)

where

$$k^2 = \frac{\omega^2}{c^2} n^2.$$ (A23)

We can write

$$\nabla \psi = ik \psi \, si \nabla \psi^* = -ik \psi^*,$$ (A24)

$$\vec{j} = i\omega c^2 ik \left(-\psi \psi^* - \psi^* \psi \right).$$ (A25)

Finally we obtain the expression of probability of current density for photons of frequency ω in a dielectric medium with refractive index n

$$\vec{j} = 2\omega^2 cn |\psi_0|^2.$$ (A26)

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