Romanian Reports in Physics, Vol. 64, Supplement, P. 1171–1177, 2012

Dedicated to Professor Ioan-Iovitz Popescu's 80th Anniversary

FRACTIONAL EULER-LAGRANGE EQUATION OF CALDIROLA-KANAI OSCILLATOR

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Received September 12, 2012

Abstract. A study of the fractional Lagrangian of the so-called Caldirola-Kanai oscillator is presented. The fractional Euler-Lagrangian equations of the system have been obtained, and the obtained Euler-Lagrangian equations have been studied numerically. The numerical study is based on the so-called Grünwald-Letnikov approach, which is power series expansion of the generating function (backward and forward difference) and it can be easy derived from the Grünwald-Letnikov definition of the fractional derivative. This approach is based on the fact, that Riemman-Liouville fractional derivative is equivalent to the Grünwald-Letnikov derivative for a wide class of the functions.

Key words: Riemann-Liouville derivatives, Caldirola-Kanai oscillator, Grünwald-Letnikov approach.

1. INTRODUCTION

Hamiltonian formulation plays an important role in classical and quantum mechanics [1, 2]. As an example the dissipative as well as the non-conservative systems can be constructed and treated using the Lagrangian and the Hamiltonian formulations [3]. Usually, dissipative systems are ascribed as having a microscopic nature [4–8]. Damped harmonic oscillator was investigated by Caldirola and Kanai [4–5]. The damped quantum harmonic oscillation with one or two degree of freedom in the framework of Caldirola-Kanai oscillator can be considered as one of the basic models of dissipation [8–9].

When studying the damped quantum harmonic oscillator in the canonical approach, two different Hamiltonian representations have been introduced. In the first representation a one-dimensional system with an exponentially increasing mass was considered (*i.e.*, the so-called the Caldirola-Kanai oscillator) [10–19]. On the other hand, the second representation the so-called Bateman Feshbach-Tikochinsky oscillator, which consists of a damped and an amplified oscillator [12, 16–19] is considered. The Caldirola-Kanai oscillator is an open system whose parameters such as mass and frequency are all time dependent, while the Bateman Feshbach-Tikochinsky oscillator is a closed system whose total energy is conserved and the dissipated energy from the damped oscillator is transferred to amplified one.

Recently, fractional calculus has found many applications in many branches of science and engineering [20–22]. Fractional differential equations have been treated numerically and analytically in many papers [21–29]. Decomposition method is one of the methods used to solve such equations numerically and analytically [30–34]. Recently a new method called matrix approach has been introduced and used [35, 36]. On the other hand, analytic solutions to fractional-order deferential equations are often expressed in terms of the Mittag-Leffler function [37–39].

In this paper, we pay attention to study numerically the fractional Euler-Lagrange equation of the so-called Caldirola-Kanai oscillator. This work is organized as follows. In section 2 we discussed briefly the basic definitions of the fractional derivatives. In section 3 we presented our model. In section 4 numerical results of the obtained Euler-Lagrange equation of the model are depicted. Finally, the concluding remarks are illustrated.

2. BASIC DEFINITIONS

In this section some fundamental formulas of fractional calculus are presented. The first one is the left Riemann-Liouville fractional integral defined as follows [20, 21]

$$ {}_aI_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) \mathrm{d}\tau \, . \tag{1} $$

The form of the right Riemann-Liouville fractional integral is

$$ {}_tI_b^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau) \mathrm{d}\tau \, . \tag{2} $$

The expression of the left Riemann-Liouville fractional derivative reads

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}}\mathrm{d}\tau . \tag{3} $$

The right Riemann-Liouville fractional derivative is given by

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{f(\tau)}{(\tau-x)^{\alpha-n+1}}\mathrm{d}\tau . \tag{4} $$

Here α is the order of the derivative such that $n-1 \leq \alpha \leq n$ and is different to zero. If α is an integer, these derivatives become the classical ones. The generalized Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} , \tag{5} $$

such that $\alpha \succ 0$, and $\beta \succ 0$. Thus, the exponential function is a special case for Mittag-Leffler function, namely

$$ E_{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) . \tag{6} $$

3. THE MODEL

We start our formalism by considering a harmonic oscillator whose mass depends on time such that $m(t) = m\exp(s\sin\beta\gamma t)$, and described by the following Hamiltonian:

$$ H = \frac{p^2}{2m}\exp(-\sin\beta\gamma t) + \frac{m}{2}q^2\exp(\sin\beta\gamma t) . \tag{7} $$

The mass depends explicitly on time, β, γ are variable parameter and damping factors, while p, and q are canonical conjugate. If $\exp(s\sin\beta\gamma t)$ is Taylor expanded to first order in increasing power of $\beta\gamma t$ with $\beta \to 1$. Then, Eq. (7) is reduces to Caldirola-Kanai Oscillator.

The Lagrangian corresponding to the Hamiltonian given by Eq. (7) is given as:

$$ L = \exp(s\sin\beta\gamma t)[\frac{1}{2}m\dot{q}2 - \frac{1}{2}mw^2(t)q^2] . \tag{8} $$

The classical equation of motion is that of a damped oscillator

$$ \ddot{q}(t) + \beta\gamma\cos\beta\gamma t\dot{q}(t) + w^2(t)q(t) = 0 . \tag{9} $$

The fractional counterpart of Eq. (8) can be written as:

$$L^F = E_{1,1}(s\mathrm{in}\beta\gamma t)[\frac{1}{2}m({}_aD_t^\alpha q)^2 - \frac{1}{2}mw^2(t)q^2]. \qquad (10)$$

Using the general form for the fractional Euler-Lagrange equation, namely

$$\frac{\partial L}{\partial q_i} + {}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q_i} + {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q_i} = 0, \qquad (11)$$

the fractional Euler-Lagrange equations have the following form

$${}_tD_b^\alpha (E_{1,1}(s\mathrm{in}\beta\gamma t)\, {}_aD_t^\alpha q(t)) - E_{1,1}(s\mathrm{in}\beta\gamma t)w^2(t)q(t) = 0. \qquad (12)$$

Now, our goal is to solve Eq. (12) numerically for different values of α.

4. NUMERICAL ANALYSIS

For numerical solution of the linear fractional-order equations (12) we can use the decomposition to its canonical form with substitutions $q(t) \equiv x_1(t)$ and ${}_aD_t^\alpha q(t) \equiv x_2(t)$. As a result, we obtain the set of equation in the following form:

$$\begin{aligned} &{}_aD_t^\alpha x_1(t) = x_2(t), \\ &{}_tD_b^\alpha (E_{1,1}(\sin\beta\gamma t)x_2(t)) = E_{1,1}(\sin\beta\gamma t)w^2(t)x_1(t). \end{aligned} \qquad (13)$$

Instead left and right side Riemann-Liouville fractional derivatives (3) and (4) in the set of equations (13) can be used the left and right Grünwald-Letnikov derivatives, which are equivalent to the Riemann-Liouville fractional derivatives for a wide class of the functions [40]. The Grünwald-Letnikov derivatives can be defined by using upper and lower triangular strip matrices (Podlubny's matrix approach) or we can directly apply the formula derived from the Grünwald-Letnikov definitions, backward and forward, respectively, for discrete time step *kh*, *k* = 1,2,3, … Le us consider the second approach, which works very well for linear as well as for nonlinear fractional differential equations [41]. Time interval [*a, b*] is discretized by (*N* + 1) equal grid points, where $N = (b-a)/h$. Thus, we obtain the following formula for discrete equivalents of left and right fractional derivatives:

$${}_aD_t^\alpha x_k = h^{-\alpha}\sum_{i=0}^{k} c_i x_{k-i}, \quad k = 0,...,N, \qquad (14a)$$

$${}_tD_b^\alpha x_k = h^{-\alpha}\sum_{i=0}^{N-k} c_i x_{k+i}, \quad k = N,...,0, \qquad (14b)$$

respectively, where $x_k \approx x(t_k)$ and $t_k = kh$. The binomial coefficients c_i, i = 1,2,3,…, can be calculated according to relation

$$c_i = \left(1 - \frac{1+\alpha}{i}\right) c_{i-1} \tag{15}$$

for $c_0 = 1$. Then, general numerical solution of the fractional linear differential equation with left side derivative in the form

$$ {}_a D_t^{\alpha} x(t) = f(x(t), t) \tag{16}$$

can be expressed for discrete time $t_k = kh$ in the following form:

$$x(t_k) = f(x(t_k), t_k) h^{\alpha} - \sum_{i=m}^{k} c_i x(t_{k-i}), \tag{17}$$

where m = 0 if we do not use a short memory principle, otherwise it can be related to memory length. Similarly we can derive a solution for an equation with right side fractional derivative. Let us consider the different value of order α for simulation time 1 second and time step h = 0.0005. The parameters set up are the following: β = 1, γ = 10, and initial condition $q(0)$ = 0.1 and $w(t) = 2t$.

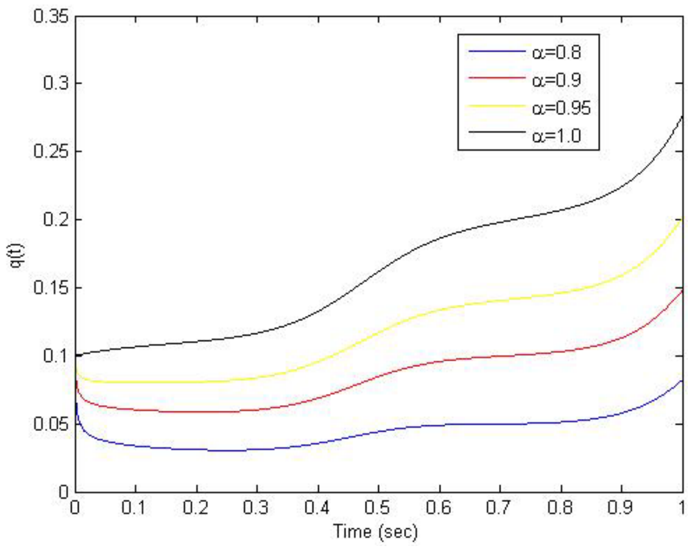

Fig. 1 – Numerical solution of equation (18) for time 1 second.

Let us consider the different function $w(t)$ in the form $w(t) = 4t + 1$ and parameters: β = 1, γ = 10, and initial condition $q(0)$ = 0.1.

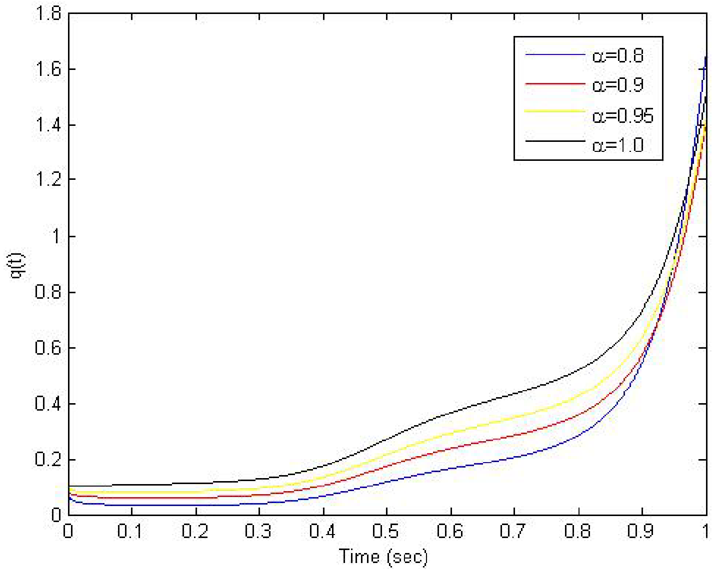

Fig. 2 – Numerical solution of equation (18) for time 1 second.

In Fig. 1 and Fig. 2 are depicted the simulation results of equations (17) for parameters $\beta = 1$, $\gamma = 10$, and various order α, where derivative interval is $a = 0$ and $b = 1$, initial condition $q(0) = 0.1$, for total simulation time 1 second and computational time step $h = 0.0005$.

5. CONCLUSIONS

In this paper we investigated the numerical solutions of the fractional Caldirola-Kanai Euler-Lagrange equations. We started with the classical Caldirola-Kanai Lagrangian and then we fractionalized it and we obtained the fractional Euler-Lagrange equations. Finally, we investigated numerically the solution of the fractional Euler-Lagrange equations obtained. Our numerical method used is based on the formula derived from the Grünwald-Letnikov definitions, backward and forward, respectively. The numerical results are shown in Fig. 1, and Fig. 2. For example, Fig. 1 shows the numerical solution of Eq. (17) for parameters $\beta = 1$, $\gamma = 10$, and various order α, and initial condition $q(0) = 0.1$ and $w(t) = 2t$. While, Fig. 2 shows the numerical solution of Eq. (17) for parameters $\beta = 1$, $\gamma = 10$, and various order α, initial condition $q(0) = 0.1$, and $w(t) = 4t + 1$.

It is clear from the figures that the behaviors of the fractional Euler-Lagrange equation strongly depend on the order of the fractional derivative, in addition to the form of the function $w(t)$. For each graph we provided the classical solution of the equations ($\alpha = 1$) and three different cases for α.

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