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# DISCRETE GINZBURG-LANDAU SPATIOTEMPORAL OPTICAL SOLITONS: COLLISION SCENARIOS

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*Abstract*. I overview recent systematic results of collisions between discrete Ginzburg-Landau spatiotemporal optical solitons (discrete Ginzburg-Landau light bullets) in both one-dimensional waveguide arrays and in two-dimensional (2D) photonic lattices in the presence of gain and loss. Depending on the numerical values of the the kick (collision momentum) and of the nonlinear (cubic) gain, the following generic collision scenarios were put forward: (a) soliton merging, (b) creation of an extra soliton (soliton birth), (c) soliton bouncing, (d) soliton spreading, and (e) quasi-elastic (symmetric) interactions.

*Key words:* spatiotemporal optical solitons, Ginzburg-Landau solitons, discrete solitons, collision scenarios.

## 1. INTRODUCTION

After the theoretical prediction [1, 2] and the subsequent experimental demonstration [3, 4] of nonlinearity-induced light localization near the edge of one-dimensional nonlinear waveguide arrays, which can lead to the formation of the so-called discrete surface solitons, the interest in the study of nonlinear self-trapped optical surface waves has been grown in recent years (for earlier works in this area, see, Refs. [5]–[8]).

The unique features of such discrete surface optical solitons in other relevant physical settings have been recently investigated both theoretically [9]–[12] and experimentally [13]–[16] (see also Ref. [17] for recent comprehensive overviews of experimental and theoretical developments in the area of discrete optical solitons).

The concept of spatial discrete surface solitons has been extended to spatiotemporal discrete surface solitons, too [18]–[21]; these discrete localized spatiotemporal structures (discrete light bullets) [22]–[23] are properly described by continuous-discrete nonlinear partial differential equations similar to those investigated earlier for cubic [24]–[26] and quadratic [27] nonlinear optical media.

The presence of gain and loss (due to optical amplifiers and saturable absorbers) in periodic photonic structures such as one-dimensional waveguide arrays and two-dimensional periodic photonic structures would influence the unique properties of both discrete surface and bulk solitons. It is well known that discrete dissipative solitons are possible in both one- and two-dimensional photonic lattices [28]–[32] and, as a result of discreteness, the dissipative physical systems exhibit novel features that have no counterpart in either the continuous limit or in other conservative discrete systems.

We recently considered continuous-discrete spatiotemporal models described by the complex Ginzburg-Landau (GL) equation [33, 34]. Thus the presence of gain and loss due to optical amplifiers and saturable absorbers in truncated one- and two-dimensional periodic photonic structures has been investigated and dissipative surface light bullets (dissipative surface spatiotemporal solitons) were introduced in both one-dimensional waveguide arrays [33] and in two-dimensional photonic lattices [34]. Similar to other types of discrete dissipative solitons in both one- and two-dimensional lattices [28, 32], the dissipative surface light bullets exhibit novel features that, as a result of both discreteness and gain (loss) effects, have no counterpart in either the continuous limit or in other conservative discrete models for both cubic and quadratic nonlinear media [35]–[40].

The GL equation is a ubiquitous model in many physical problems [41], and in different forms it appears as the simplest model for describing dissipative solitons in different settings [42]–[46], clusters of localized states rotating around a central vortex core [47], and laser patterns in cavities [48, 49]. In application to waveguide arrays, a periodic refractive index modulation can be modeled by a discrete GL equation that describes the presence of gain and loss due to optical amplifiers and saturable absorbers [28].

Recently, we found the domains of existence and stability of such dissipative light bullets in the relevant parameter space, for both on-site and inter-site solitons and for the states localized at different distances from the edge of the waveguide array or in the corners, at the edges, and in the center of two-dimensional photonic lattices [33, 34]. Once stable dissipative discrete spatiotemporal optical solitons (discrete GL light bullets) are available, a problem of great interest is to consider collisions between them.

In this work I briey overview the recent studies of collisions between discrete Ginzburg-Landau spatiotemporal optical solitons (“discrete Ginzburg-Landau light bullets”), which form in (i) one-dimensional waveguide arrays (i.e., in one-dimensional photonic lattices), and (ii) in two-dimensional photonic lattices. I find the generic collision scenarios of discrete Ginzburg-Landau light bullets forming both near the edges of a semi-infinite array of weekly coupled nonlinear waveguides and in the center of the waveguide array. I also identify the collision outputs in the case of discrete Ginzburg-Landau light bullets located both in the corners or at the edges of a truncated twodimensional photonic lattice and deep inside the lattice (i.e., in the center of the two-dimensional lattice).

Depending on the value of the kick parameter (collision momentum), four generic outcomes were identi_ed in the case of collision of two identical discrete GL light bullets located at equal distances from the edge of the waveguide array: (a1) merger of the discrete GL solitons into a single one, at small values of the kick parameter, (b1) creation of an extra discrete GL soliton at intermediate values of the collision momentum, (c1) quasi-elastic interactions at both intermediate values of the kick parameter (for relatively small values of the cubic gain) and at large values of the kick parameter (for relatively high values of cubic gain), and (d1) dissipative soliton spreading at relatively large values of the collision momentum but only in the case of relatively small values of the cubic gain. In the case of collision of two nonidentical discrete dissipative light bullets located at different distances from the edge of the waveguide array four generic outcomes were identified too: (e1) soliton bouncing, accompanied by a sharp modification of soliton velocities during the interaction process, for relatively small values of the collision momentum, (f1) soliton creation at intermediate values of the kick parameter and for relatively low values of the cubic gain, (g1) soliton spreading (in time) at intermediate values of the collision momentum and for relatively high values of the cubic gain, and (h1) quasi-elastic interactions at large values of the kick parameter.

In the case of discrete GL light bullets forming in two-dimensional photonic lattices, depending on the value of the kick parameter (collision momentum), four generic outcomes were identified, too. Thus collision of two identical solitons located in the corner, at the edge, and in the center of the photonic lattice leads to the following outputs: (a2) merger of the discrete GL light bullets into a single one, at small values of the kick parameter (soliton transverse velocity), (b2) creation of an extra discrete GL light bullet at intermediate values of the collision momentum, (c2) quasi-elastic (symmetric) interactions at both intermediate values of the kick parameter (for relatively small values of the cubic gain) and at large values of the kick parameter (for relatively high values of cubic gain), and (d2) soliton spreading at relatively large values of the collision momentum but only in the case of relatively small values of the cubic gain. In the case of collision of two non-identical corner and edge discrete GL light bullets located at different distances from the edges of the two-dimensional photonic lattice four generic outcomes were identified, too: (e2) soliton bouncing, accompanied by a sharp modification of soliton velocities during the interaction process, for relatively small values of the collision momentum, (f2) soliton creation at intermediate values of the kick parameter and for relatively low values of the cubic gain, (g2) soliton spreading (in time) at intermediate values of the collision momentum and for relatively high values of the cubic gain, and (h2) quasi-elastic (symmetric) interactions at relatively large values of the kick parameter.

## 2. DISCRETE GINZBURG-LANDAU LIGHT BULLETS IN ONE-DIMENSIONAL WAVEGUIDE ARRAYS: COLLISION SCENARIOS

A nonlinear system of coupled GL equations describes, in the framework of the tight-binding approximation of the coupled-mode theory, the spatiotemporal dynamics of semi-infinite waveguide arrays in the presence of gain and loss [33]. We restricted ourselves to the case of anomalous GVD, self-focusing nonlinearity and in-phase (unstaggered) solitons. Classes of stable on-site centered continuous-discrete spatiotemporal solitons of the coupled discrete GL equations were thoroughly investigated [33].

Once stable discrete GL light bullets were available, a problem of great interest is to consider collisions between them. We recently investigated this issue in the framework of the nonlinear coupled continuous-discrete GL evolution equations. We found the generic outcomes of collisions between discrete light bullets located at different distances $d = 0$, 1, and 20 from the edge of semi-infinite waveguide arrays.

We took a pair of stable discrete GL light bullets separated by a large temporal distance $t_2 - t_1 = T$, and we solved the continuous-discrete GL equations with the initial condition (at $z = 0$) corresponding to collisions between two stable discrete dissipative solitons: $E_n(t, 0) = E_n(t + T/2) \exp(i \chi t) + E_n(t - T/2) \exp(-i \chi t)$, where $E_n(t, 0$ is the shape of the electric field at $z = 0$, and $\chi$ is the kick parameter (collision momentum).

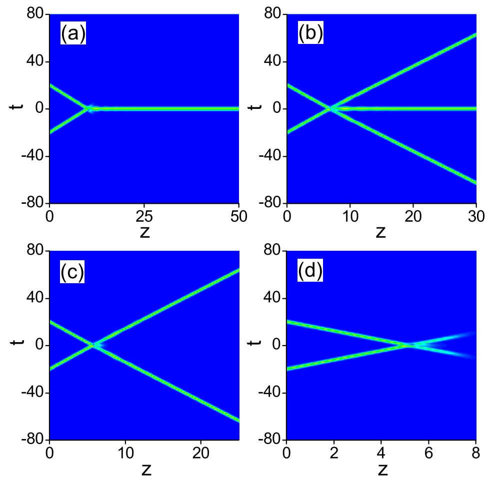


Fig. 1 – Contour plots display the evolution of the electric field in the plane (t, z) for on-site discrete Ginzburg-Landau light bullets localized at the edge of the waveguide array (d = 0). Here ε = 3.5, and the kick parameter χ = 2 (a), χ = 3 (b), χ = 3.6 (c), and χ = 4 (d).

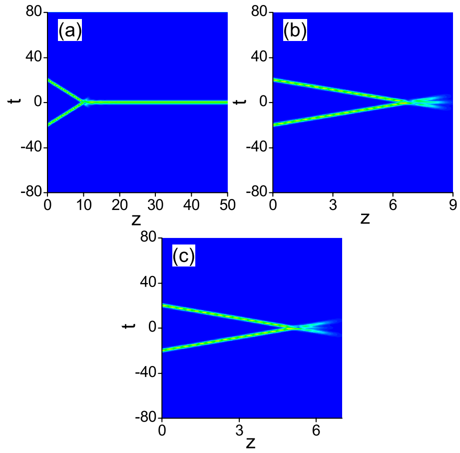

Fig. 2 – Same as in Fig. 1 but for on-site discrete Ginzburg-Landau light bullets localized at the distance d = 1 from the edge of the waveguide array. Here ε = 3.5, and the kick parameter χ = 2 (a), χ = 3 (b), and χ = 4 (c).

The systematic simulations of collisions between two identical dissipative discrete spatiotemporal optical solitons initially set at equal distances $d_1 = d_2 = 0$, 1, and 20 from the edge of the truncated one-dimensional waveguide array have shown

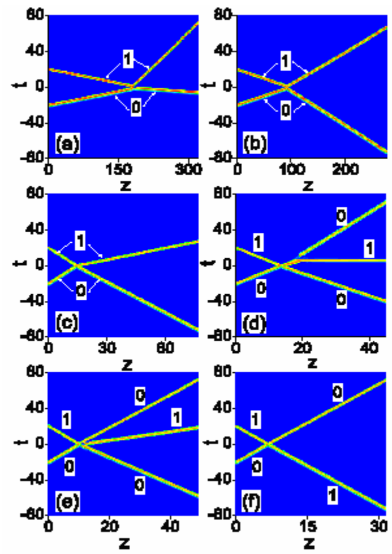

Fig. 3 – Soliton trajectories corresponding to the collision scenarios of onsite discrete Ginzburg-Landau light bullets localized at d = 0 and d = 1. Here ε = 3.4, μ = 1, and the kick parameter χ = 0.1 (a), χ = 0.2 (b), χ = 1.4 (c), χ = 1.5 (d), χ = 2 (e), and χ = 3 (f).

that by gradually increasing the initial kick parameter χ, the following four generic outcomes have emerged (see Figs. 1–4) [50]:

(a) Merger of the two identical dissipative discrete light bullets into a single one at small values of the kick parameter $\chi$;

(b) Creation of an extra dissipative discrete light bullet at intermediate values of the kick parameter $\chi$;

(c) quasi-elastic interactions at both intermediate values of $\chi$ (for relatively small values of the cubic gain $\varepsilon$) and at large values of $\chi$ (for relatively high values of the cubic gain $\varepsilon$);

(d) Spreading of the dissipative discrete solitons at relatively large values of the kick parameter $\chi$ but only in the case of relatively small values of nonlinear (cubic) gain parameter.

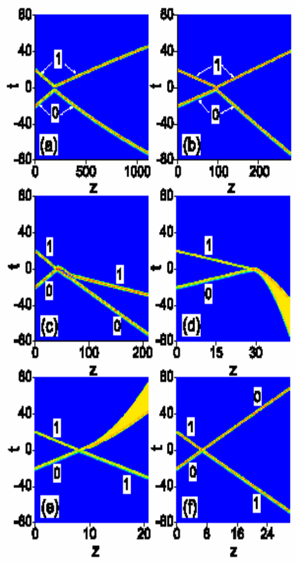


Fig. 4 – The same as in Fig. 3 but for $\varepsilon = 3{:}6$. Here the kick parameter $\chi = 0.1$ (a), $\chi = 0.2$ (b), $\chi = 0.5$ (c), $\chi = 0.7$ (d), $\chi = 2.5$ (e), and $\chi = 3$ (f).

The merger of the two colliding solitons into a single one, a promising effect for potential applications in photonics has been reported before in other relevant physical settings: (a) the study of solitons in saturable materials with a linear and quadratic intensity depending refraction index change [51], (b) the study of dynamics and collisions of moving solitons in the standard model of fiber gratings [52] and in Bragg gratings with dispersive reflectivity [53], (c) the study of collisions of both nonspinning and spinning three-dimensional GL solitons [54], and in the study of collisions between discrete surface light bullets in nondissipative one- and two-dimensional photonic lattices [55].

We mention also that the process of interaction between two stable three-dimensional dissipative light bullets, separated either in time or in space [56], revealed the importance of the input phase difference between them. By controlling this key collision parameter, soliton merging (fusion), erasure of one of the colliding solitons, and the formation of double light bullet complexes (i.e., the formation of light bullet "molecules") were put forward [56].

It is worthy to emphasize that the transformation of two colliding solitons into three, one quiescent and two moving has also been reported in collisions of solitons in media with saturable nonlinearity [51], collisions of one-dimensional dissipative spatial solitons in periodically patterned semiconductor amplifiers [57], collisions of coaxial three-dimensional spatiotemporal GL solitons [54], and collisions of moving solitons in Bragg gratings with dispersive reflectivity [53].

## 3. DISCRETE GINZBURG-LANDAU LIGHT BULLETS IN TWO-DIMENSIONAL PHOTONIC LATTICES: COLLISION SCENARIOS

Recently we considered light propagation and continuous-discrete spatiotemporal soliton formation in a nonlinear square dissipative photonic lattice (i.e., in a nonlinear two-dimensional photonic lattice) created by weakly coupled arrays of identical evenly spaced two-dimensional homogeneous waveguides [34]. We worked in the framework of the standard coupled-mode approach where only the amplitude $E_{n,m}$ of the electric field in the lattice site ($n$, $m$) was considered to evolve during propagation while the mode profile was assumed to remain constant. Therefore, the slowly varying normalized envelope $E_{n,m}$ obeys a set of coupled partial differential equations (continuous-discrete GL coupled equations) describing the spatiotemporal dynamics of light in a two-dimensional photonic lattice in the presence of gain and loss [34].

A problem of great interest is to consider collisions between discrete spatiotemporal dissipative solitons (discrete spatiotemporal GL solitons). Recently, we investigated this issue in the framework of the nonlinear coupled continuous-discrete Ginzburg-Landau evolution equations focusing on finding the generic outcomes of collisions between discrete dissipative light bullets located in the corner, at the edge and in the center of the two-dimensional photonic lattice [58].

Next we briey discuss the outcomes of collisions between two identical corner, edge and center spatiotemporal GL solitons. In a recent paper [58] we found that by gradually increasing the initial kick (collision momentum) $\chi$, we got the following four generic collisions outcomes, which are illustrated in Figs. 5–7:

(a) *Merging* of the two identical GL solitons into a single one at small values of the kick parameter $\chi$;

(b) *Creation* of an extra GL soliton at intermediate values of the kick parameter $\chi$;

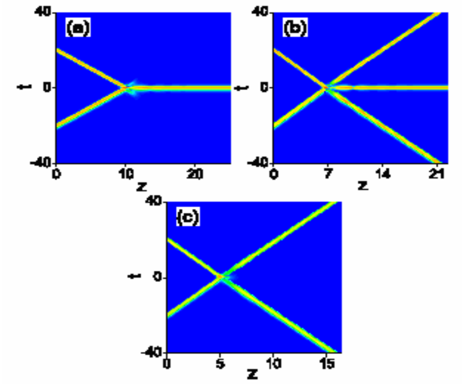


Fig. 5 – The (0, 0) corner discrete Ginzburg-Landau light bullet trajectories showing soliton fusion (a), soliton birth (b), and elastic collision (c). Here ε = 3.6, and the collision momentum χ = 2 (a), χ = 3 (b), and χ = 4 (c).

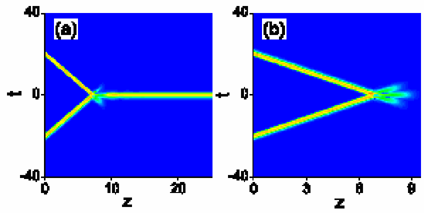


Fig. 6 – Illustration of the soliton trajectories of the (20, 20) center discrete Ginzburg-Landau light bullets for the soliton fusion scenario (a) and the soliton decay scenario (b). Here μ = 1 and ε = 3.6. The kick parameters are χ = 2.8 (a) and χ = 2.9 (b).

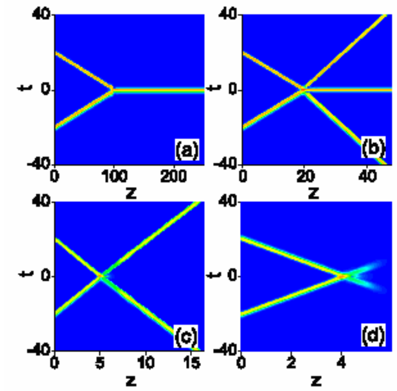


Fig. 7 – Trajectories of the (20, 20) center discrete Ginzburg-Landau light bullets, which correspond to soliton fusion (a), soliton birth (b), ellastic collision (c), and soliton decay (d). Here ε = 3.7, and the kick parameter χ = 0.2 (a), χ = 1 (b), and χ = 4 (c), and χ = 5 (d).

(c) *Quasi-elastic (symmetric) interactions* at both intermediate values of χ (for relatively small values of the cubic gain ε) and at large values of χ (for relatively high values of the cubic gain ε);

(d) Spreading of the GL solitons at relatively large values of the kick parameter χ but only in the case of relatively small values of nonlinear (cubic) gain parameter.

We also obtained in Ref. [58] the collision outputs when the two nonidentical input solitons are located at different sites of the two-dimensional photonic lattice. We thus considered in detail the collisions between (0, 0) and (1, 1) corner solitons and between (20, 0) and (20, 1) edge solitons for two representative values of the nonlinear (cubic) gain parameter, i.e., for both relatively small values and relatively large values of this key parameter of the discrete dissipative system.

Thus by increasing the initial kick parameter χ (transverse velocity), we got the following four generic outcomes [58], which are illustrated in Figs. 8–9 by contour plots of the *z*-dependence of the field evolution in the plane (*t*, *z*):

(e) *Soliton bouncing*, accompanied by a sharp modification of the soliton velocities during the interaction process, for relatively small values of χ;

(f) *Soliton creation* (*soliton "birth"*) at intermediate values of χ and for relatively low values of the cubic gain parameter ε;

(g) *Soliton spreading (in time)* at intermediate values of χ and for relatively high values of the cubic gain ε;

(h) quasi-elastic (symmetric) interactions at relatively large values of χ.

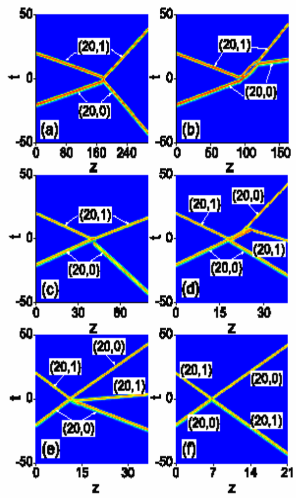

Fig. 8 – Trajectories corresponding to collisions between (20, 0) and (20, 1) edge discrete Gizburg-Landau light bullets. Here ε = 3.5 and the collision parameter χ = 0.1 (a), χ = 0.2 (b), χ = 0:5 (c), χ = 1.1 (d), χ = 1.8 (e), and χ = 3 (f).

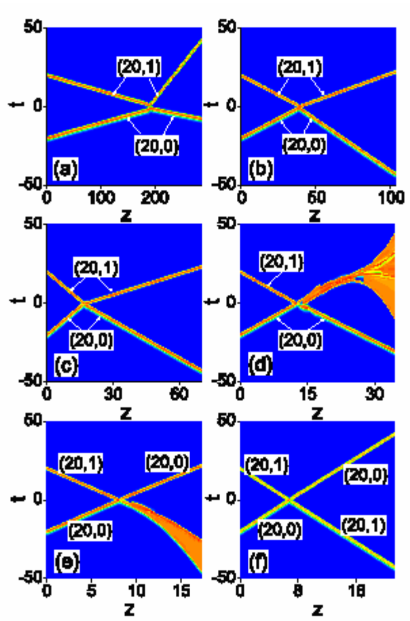


Fig. 9 – The same as in Fig. 8 but for ε = 3.7. Here the kick parameter is $\chi = 0.1$ (a), $\chi = 0.5$ (b), $\chi = 1.2$ (c), $\chi = 1.6$ (d), $\chi = 2.5$ (e), and $\chi = 3$ (f).

## 4. CONCLUSIONS

In this paper I have overviewed some recent results concerning collisions of discrete Ginzburg-Landau spatiotemporal optical solitons ("discrete dissipative light bullets"). I have considered discrete dissipative light bullets forming in both one-dimensional arrays of weekly coupled nonlinear waveguides and in two-dimensional photonic lattices. These dissipative localized structures are accurately described by cubic-quintic complex Ginzburg-Landau coupled equations accounting for gain and loss due to optical amplifiers and saturable absorbers.

These dissipative nonlinear photonic lattices can support continuous-discrete dissipative spatiotemporal solitons (dissipative light bullets), which are stable in a broad region of their parameters. I have described several generic collision scenarios occuring in such physical settings: (a) soliton merging, (b) creation of an extra soliton (soliton birth), (c) soliton bouncing, (d) soliton spreading, and (e) quasi-elastic (symmetric) interactions. I do believe that these generic collision outcomes can be found for other types of discrete dissipative solitons.

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