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MANDELBROT SCALING AND PARAMETRIZATION INVARIANT THEORIES

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Abstract. Fractional variational principles have gained considerable importance during the last decade due to their applications in several areas of sciences and engineering. In this paper we will adapt this variational principle to obtain the Euler-Lagrange equation of motion, by considering two different cases. In the first case we used the scaling concepts of Mandelbrot of fractals in variational problems of mechanical systems in order to re-write the action function as an integration over a scaling measure. After that we parameterize the time in the action integral to obtain the equations of motion. It is shown that the genuine Euler-Lagrange equations of motion are those which are obtained using the Mandelbrot scaling of space/and or time.

Key words: fractional dimensional space, variational principle.

1. INTRODUCTION

As it is known Hausdorff introduced the notion of fractional dimension. This concept became very important especially after the revolutionary discovery of fractal geometry by Mandelbrot [1], where he used the concept of fractionality and worked out the relations between fractional dimension and integer dimension by using the scale method i.e. $\mathrm{d}^{\alpha}x = \dfrac{\pi^{\alpha/2}|x|^{\alpha-1}}{\Gamma(\alpha/2)}\mathrm{d}x, \quad 0<\alpha\le 1$. We mention that various efforts in this direction has been made by many researchers in several branches of science and technology [2-18].

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Besides, there are other approaches to describe fractional dimension. These include, fractional calculus which represents a generalization of differentiation and integration to non integer order [see for example the following references 19–22] and the analytic continuation of the dimension in Gaussian integral [12, 23–25]. The later is often used in quantum field theory [24, 25], and introduced in the dimensional regularization method. The fractional calculus provide a possible calculus to deal with fractals. As a result this leads to a characteristic problem to study the fractional differentiability properties of nowhere differentiable functions, and, to investigate a possible relation of the order of differentiability with the dimension of the graph of the function.

In reference [26], we used the scaling concepts of Mandelbrot to obtain the equations of motion of mechanical systems. Such treatment needs careful when applying the scaling method. The natural question which arises here is that: In the following action function, $S(q)=\int_a^b L(q_i(\tau),\dot{q}_i(\tau))\,\mathrm{d}\tau,\quad i=1,2,\ldots,n,$ should we scale the measure of integration $\mathrm{d}\tau$?, or should we scale, both the measure $\mathrm{d}\tau$ and velocities $\dot{q}_i(\tau)$?. The answer is that, if use the first approach then this will coincide with Mandelbrot concepts of scaling the space-time dimensions and the Euler-Lagrange equations of motion are the genuine ones. On the other hand, if use the parameterization method, then the new equations of motion are identical with the original equations of motion and this is nothing but a change of variables.

The plan of this paper is as follows:

In section 2 we present the Euler-Lagrange equations of motion by considering both the Mandelbrot scaling method (MSM) and the parametrization of the time. In section 3 we solve two discrete systems by the two methods. In section 4 we present the Euler-Lagrange equations of motion for field systems in fractional dimensional and we give an example from fractal solids. Finally, section 5 presents the conclusions.

2. TIME SCALING VARIATIONAL METHOD (TSVM)

In this section we will derive the Euler-Lagrange equations of motion by considering both Mandelbrot scaling method (MSM) and the parametrization of the time. Let us consider the action function of the form

$$S(q)=\int_a^b L(q_i(t),\dot{q}_i(t))\,\mathrm{d}\tau,\quad i=1,2,\ldots,n. \tag{1}$$

Using the scaling of time

$$\mathrm{d}^{\alpha}t=f(\alpha)|t|^{\alpha-1}\,\mathrm{d}t, \tag{2}$$

where $f(\alpha)=\dfrac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}$ and $0<\alpha\le 1$, the fractional action function can be written as

$$S^{\alpha}(q)=\int_a^b L\big(q_i(\tau),\dot{q}_i(\tau),\tau\big)\,\mathrm{d}^{\alpha}\tau,\quad i=1,2,\ldots,n, \tag{3}$$

where $a<\tau<b$. Here we note that when $\alpha\to 1$, the functional $S^{\alpha}\big(q(t)\big)$ is just the well known classical action function in classical mechanics.

Using the principles of calculus of variation, and after putting the variation $\delta S^{\alpha}=0$, we obtain the modified classical equations of motion as

$$\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\alpha-1}{\tau}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0. \tag{4}$$

Now if we parameterize the time as

$$t\to\frac{f(\alpha)}{\alpha}\tau^{\alpha}, \tag{5}$$

where, $0<\alpha\le 1$ then the action function can be written as

$$S^{\alpha}(q)=\int_a^b L^{*}\big(q_i(\tau),\dot{q}_i(\tau),\tau\big)\,\mathrm{d}^{\alpha}\tau,\quad i=1,2,\ldots,n, \tag{6}$$

where $\dot{q}_i(t)\to\dot{q}_i(\tau)\dfrac{\tau^{1-\alpha}}{f(\alpha)}$ and L^{*} is the corresponding Lagrangian. In this case the Euler-Lagrange equations of motion read as

$$\frac{\partial L^{*}}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L^{*}}{\partial \dot{q}_i}\right)-\frac{\alpha-1}{\tau}\left(\frac{\partial L^{*}}{\partial \dot{q}_i}\right)=0. \tag{7}$$

3. ILLUSTRATIVE EXAMPLES

As a first example of the TSVM discussed in previous section let us consider the following Lagrangian

$$L=\frac{1}{2}\dot{x}^{2}(t). \tag{8}$$

Using the MSM, the fractional action function is given by

$$S^{\alpha}(x)=\int_a^b\left[\frac{1}{2}\dot{x}^{2}(\tau)\right]\mathrm{d}^{\alpha}\tau, \tag{9}$$

and the Euler-Lagrange equation of motion is obtained as

$$\tau^2 \ddot{x} + (\alpha - 1)\dot{x} = 0. \tag{10}$$

Eq. (10) has the following solution

$$x(\tau) = c\tau^{2-\alpha}, \tag{11}$$

where c is a constant. Now parameterize the time, then the action function is given by

$$S^\alpha(x) = \int_a^b \left[\frac{1}{2}\tau^{2-\alpha} \frac{\dot{x}^2(\tau)}{f^2(\alpha)} \right] \mathrm{d}^\alpha \tau. \tag{12}$$

The Euler-Lagrange equations of motion is obtained as

$$\tau^2 \ddot{x} + (1 - \alpha)\dot{x} = 0. \tag{13}$$

Eq. (13) has the following solution

$$x(\tau) = c\tau^\alpha, \tag{14}$$

It is obvious that this solution is nothing but the solution of a free particle if we use the change of variable $t \to \frac{f(\alpha)}{\alpha}\tau^\alpha$.

As a second example, let us consider a simple pendulum of length l attached to the circumference of a body of negligible radius and mass m. The Lagrangian for this system is given as

$$L = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2} mgl\theta^2. \tag{15}$$

Here θ denotes the angular coordinate. Using the Mandelbrot scaling method and its variational principle, the Euler-Lagrange equation of motion is calculated as

$$\ddot{\theta} + \frac{(\alpha - 1)}{\tau}\theta + mgl\theta = 0. \tag{16}$$

The solution of Eq. (16) is given by

$$\theta(\tau) = C_1(\tau)^{1-\frac{\alpha}{2}} J\left(\frac{\alpha}{2} - 1, \sqrt{mgl\tau}\right) + C_2(\tau)^{1-\frac{\alpha}{2}} Y\left(\frac{\alpha}{2} - 1, \sqrt{mgl\tau}\right), \tag{17}$$

where J and Y are the modified Bessel functions of the first and second kind respectively. Now if we parameterize the time, then the modified Lagrangian is given as

$$L^* = \frac{1}{2f^2(\alpha)}\dot{\theta}^2\tau^{2-2\alpha} - \frac{1}{2}mgl\theta^2. \tag{18}$$

The Euler-Lagrange equations of motion is obtained as

$$\tau^2\ddot{\theta} + (1-\alpha)\tau\dot{\theta} + \omega_\alpha^2\theta\tau^{2\alpha} = 0, \tag{19}$$

where $\omega_\alpha = \frac{1}{\alpha}\sqrt{\frac{gf(\alpha)}{l}}$. The solution of Eq. (19) is calculated as

$$\theta(\tau) = A\cos(\tau^\alpha\omega_\alpha). \tag{20}$$

If we use the change of variables $t \to \frac{f(\alpha)}{\alpha}\tau^\alpha$, then the solution (20) is simply the standard simple pendulum solution as

$$\theta(t) = A\cos(\omega_1 t). \tag{21}$$

4. EULER-LAGRANGE EQUATIONS OF MOTION FOR FIELD SYSTEMS IN FRACTIONAL DIMENSIONAL SPACE

The possible extension for the above TSVM can be generalized for field systems. A covariant form of the action would involve a Lagrangian density $\mathcal{L}$ via $S = \int_{\partial\Omega'} \mathcal{L}\mathrm{d}^{D+1}x = \int \mathcal{L}\mathrm{d}^D x\mathrm{d}t$, where $\partial\Omega'$ is the boundary for all coordinates. The Lagrangian density $\mathcal{L}$ is defined as, $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ and with $L = \int \mathcal{L}\mathrm{d}^D x$. The corresponding covariant Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \tag{22}$$

where ϕ is the field variable and ∂_μ is space and time derivative.

For non-integer space-time coordinates, the action function for N degrees of freedom is

$$\begin{aligned} S &= \int_{\partial\Omega'} \mathrm{d}^{D_t}t\,\mathrm{d}^{D_s}x\,\mathcal{L}(\phi, \partial_\mu\phi), \\ &= \int \mathrm{d}^{\alpha_t}t \int \prod_{i=1}^{N} \mathrm{d}^{\alpha_i}x_i\,\mathcal{L}(\phi, \partial_\mu\phi), \end{aligned} \tag{23}$$

where, ϕ and $\partial_{\mu}\phi$ are functions of $\left(t, x^1, \ldots, x^N\right)$ and $\partial_{\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^2}\right)$, with i is running from 1 to N and the fractional volume element $\mathrm{d}^D x$ and the fractional line element are given respectively as [1, 27]

$$\mathrm{d}^{D_s} x = \prod_i \mathrm{d}^{\alpha_i} x_i, \tag{24}$$

$$\mathrm{d}^{\alpha_i} x_i = \frac{\pi^{\alpha_i/2} |x|^{\alpha_i - 1}}{\Gamma(\alpha_i/2)} \mathrm{d}x_i, \tag{25}$$

and $D_s = \sum_{i=1}^{N} \alpha_i$, $D_t = \alpha_t$. In this paper we will consider the limits of $\alpha_{\mu} = (\alpha_t, \alpha_s)$ as $0 < \alpha_{\mu} \le 1$, such that $0 < D \le N + 1$. We set $\delta S = 0$, it follows that the Euler-Lagrange equations of motion in non-integer dimensions is given by [28]

$$\frac{\partial \mathcal{L}(\phi, \partial_{\mu}\phi)}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}(\phi, \partial_{\mu}\phi)}{\partial(\partial_{\mu}\phi)} - (\alpha_{\mu\nu} - \delta_{\mu\nu})\left(x^{(-1)}\right)^{\nu} \frac{\partial \mathcal{L}(\phi, \partial_{\mu}\phi)}{\partial(\partial_{\mu}\phi)} = 0, \tag{26}$$

with $\delta_{\mu\nu}$ is a diagonal unit matrix, $\left(x^{-1}\right) = \mathrm{column}\left(t^{-1}, \left(x^{-1}\right), \ldots, \left(x^{-N}\right)\right)$, and $\alpha_{\mu\nu}$ are the diagonal elements of a matrix which include both time and spatial dimensions $\left(\alpha = \mathrm{dimension}\left(\alpha_t, \alpha_1, \ldots \alpha_N\right)\right)$, the spatial dimension of the system is specified by $D_s = \mathrm{T}r(\alpha) - \alpha_t$.

An important point to be specified here is that we scale only the measure of integration $\mathrm{d}^D x$, without making any change of variable, in the covariant components $\left(x^{\mu}, \partial_{\mu} x^{\mu}\right)$.

To clarify the situation, let us solve the fractal porous media, in order to obtain the fractional wave equation. The Lagrangian density for a linear elastic homogenous fractal solids under small motion and zero external loads as [29].

$$\mathcal{L} = \frac{1}{2}\rho \dot{u}^2 - \frac{1}{2} E \epsilon^2, \tag{27}$$

where the strain ϵ is defined as $\epsilon = \frac{\partial u}{\partial x} = u_{,x}$. According to MSM, the Euler-Lagrange equations (26), gives the wave equation as

$$\rho\ddot{u} - E\frac{(\alpha-1)}{x}u_{,x} - Eu_{,xx} = 0. \tag{28}$$

This result is in a complete agreement with the result obtained by Tarasov [8, 9]. On the other hand, if we use the analogy of parametrization as discussed previously with the following change of variables $x \to \frac{f(\alpha)}{\alpha}x^{\alpha}$, we obtain the equation of motion as

$$\rho c_1\ddot{u} - E\left(c_1^{-1}u_{,x}\right)_{,x} = 0, \tag{29}$$

where $c_1 = f(\alpha)x^{\alpha-1}$. This is nothing but the original equation of motion $\rho\ddot{u} - Eu_{,xx} = 0$, with the change of variables $x \to \frac{f(\alpha)}{\alpha}x^{\alpha}$. Similar results for equation (29) are obtained in reference [29].

5. CONCLUSIONS

In this paper, we considered the action function using first, the concepts of Mandelbrot [13] and then parameterizing the time. It is shown that one should care when applying Mandelbrot scaling method, and the action function should be an integral over the measure of scaling quantities. The action function is obtained as an integration over fractional space-time dimensions, which gives the genuine equations of motion. Illustrative examples were solved in detail.

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