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THE ISSUE OF PRIMITIVE INDEFINITE-INTEGRAL IN THE THEORY OF FRACTAL SPACE-TIME

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Abstract. Using an interpolation method, the issue of primitive - indefinite integral in the theory of fractal space-time is established. In this context, the von Koch curve case is analyzed. Moreover, a correspondence with Cantorian space-time is determined.

Key words: fractal space-time, Cantor set, indefinite-integral.

1. INTRODUCTION

Let us consider a fractal curve F which is self affine [1, 2]. It has no derivative in none of its points. The necessity of connecting the fractal theory with the results [3-6], where the derivatives play an essential role, leads us to defining of the left and the right derivatives of a certain point of the curve. Their average values are connected with the usual derivative, while their semi-difference characterizes the fractality (for other details see [2–6]).

However, there are a number of problems in the field of Physics, where the notion of integral is also required [6-12]. This aspect will be analyzed in the followings. First, some specifications must be considered:

a) A curve I, which is the primitive of F, if there is at least in an approximation, must be continuous and derivable, hence of class C_1, since its derivative in any point should render a point of F, which is well-determined;

b) The analytical expression of I in the form $y = y(x)$ is difficult to be established. Instead of this, we can state a parameterization

$$x = x(t);\, y = y(t); \tag{1}$$

with $t \in (t_A, t_B)$ to describe a curve F between the points

$$A(x(t_A), y(t_A)) \in \mathrm{F} \text{ and } B(x(t_B), y(t_B)) \in \mathrm{F}. \tag{2}$$

As a rule, (1) could be achieved by choosing a value $t' \in (t_A, t_B)$ for an intermediary point on F between *A* and *B* and by continuing this course endlessly;

c) In the next step we shall choose an approximation of (1). For $n+1$ values of t, denoted by $t_0, t_1, ..., t_i, ..., t_n$, that satisfy

$$t_A = t_0 < t_1 < ... < t_n = t_B\,, \tag{3}$$

let us assume that the values of $x(t_i) = x_i$ and $y(t_i) = y_i$ are known. Then, we can build up the functions

$$x = X(t) \text{ and } y = Y(t) \tag{4}$$

so that

$$X(t_i) = x(t_i) \text{ and } Y(t_i) = y(t_i)\,. \tag{5}$$

The curves (4) are interpolation functions on the known values of (1) in the points (3).

2. EXAMPLES OF INTERPOLATION

There are many possibilities to obtain the interpolation functions (5) – for details see [7], - which are continuous, hence of class C_0, as well as F. Three of these possibilities are presented as follows:

a) One of them is to connect the neighboring points by a segment. Between the points with the coordinates $x(t_{i-1})$, $x(t_i)$ and $x(t_{i+1})$, $0 < i < n$ we shall have two segments whose equations are

$$\begin{cases} x(t) = x(t_{i-1}) + \dfrac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\, t, & \text{with} \quad t_{i-1} < t < t_i \\ x(t) = x(t_i) + \dfrac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}\, t, & \text{with} \quad t_i < t < t_{i+1} \end{cases}$$

with the primitives

$$\begin{cases} \int x(t)\,\mathrm{d}t = x\left(t_{i-1}\right)t + \dfrac{x\left(t_i\right) - x\left(t_{i-1}\right)}{t_i - t_{i-1}}\dfrac{t^2}{2} + C_{i-1}, \text{ with } \quad t_{i-1} < t < t_i \\ \int x(t)\,\mathrm{d}t = x\left(t_i\right)t + \dfrac{x\left(t_{i+1}\right) - x\left(t_i\right)}{t_{i+1} - t_i}\dfrac{t^2}{2} + C_i, \qquad \text{with } \quad t_i < t < t_{i+1}. \end{cases} \tag{6}$$

By imposing the continuity of these primitives, one can determine the connection between the integration constants C_{i-1} and C_i. This fact provides the existence of the derivative for t_i which coincides with the value $x(t_i)$ of the interpolation function.

Perhaps this is the closest way to the fractal representation.

b) Another possibility is to choose the Lagrange interpolation polynomials of the order *n*. To obtain such an interpolation polynomial, let us denote

$$\begin{cases} P_n(t) = a_0 + a_1 t + \ldots + a_n t^n \\ Q_n(t) = b_0 + b_1 t + \ldots + b_n t^n. \end{cases} \tag{7a,b}$$

The conditions (5) require

$$P_n(t_i) = x(t_i) \text{ and } Q_n(t_i) = y(t_i), \text{ with } i = 0, \ldots, n. \tag{8}$$

Under such circumstances, let us expand the $P_n(t)$ polynomial. We have

$$\begin{cases} a_0 + a_1 t_0 + \ldots + a_n t_0^n = x(t_0) \\ \ldots \qquad \ldots \qquad \ldots \\ b_0 + b_1 t_n + \ldots + b_n t_n^n = x(t_n), \end{cases} \tag{9}$$

which forms a system of $(n+1)$ equations for the $(n+1)$ coefficients $a_0, \ldots, a_n$ of the required polynomial. By adding (7a) to the system (9), we obtain $(n+2)$ equations for these coefficients. The compatibility condition gives

$$\begin{vmatrix} 1 & t_0 & \ldots & t_0^n & x(t_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & t_n & \ldots & t_n^n & x(t_n) \\ 1 & t & \ldots & t^n & P_n(t) \end{vmatrix} = 0,$$

i.e.

$$\begin{vmatrix} 1 & t_0 & \dots & t_0^n & x(t_0) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & t_n & \dots & t_n^n & x(t_n) \\ 0 & 0 & \dots & 0 & P_n(t) \end{vmatrix} + \begin{vmatrix} 1 & t_0 & \dots & t_0^n & x(t_0) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & t_n & \dots & t_n^n & x(t_n) \\ 1 & t & \dots & t^n & 0 \end{vmatrix} = 0,$$

and from here

$$P_n(t) = -\begin{vmatrix} 1 & t_0 & \dots & t_0^n & x(t_0) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & t_n & \dots & t_n^n & x(t_n) \\ 1 & t & \dots & t^n & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & t_0 & \dots & t_0^n \\ \dots & \dots & \dots & \dots \\ 1 & t_n & \dots & t_n^n \end{vmatrix}. \tag{10}$$

The $Q_n(t)$ polynomial is also obtained in a similar manner.

c) We can also accomplish the interpolation by the "Fourier sums", *i.e.* by a part of a Fourier's series, named trigonometrical polynomials, such as

$$T(t) = a_0 + \sum_{k=1}^{m} (a_k \cos kt + b_k \sin kt). \tag{11}$$

By taking into account that the number of the coefficients to be determined is $2m+1$, we shall require the same number of points in which the function is known. The determination methods for $T(t)$ are similar to those used in b). Thus, by imposing (11), we obtain

$$a_0 + \sum_{k=1}^{m} (a_k \cos kt + b_k \sin kt) = x(t_i),\ i = \overline{0, 2m}. \tag{12}$$

The compatibility of the equations (12) and (11) for the unknown values a_k, b_k $\left(k = \overline{0, 2m}\right)$ requires

$$\begin{vmatrix} 1 & \cos t_0 & \sin t_0 & \dots & \cos mt_0 & \sin mt_0 & x(t_0) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \cos t_{2m} & \sin t_{2m} & \dots & \cos mt_{2m} & \sin mt_{2m} & x(t_{2m}) \\ 1 & \cos t & \sin t & \dots & \cos mt & \sin mt & T(t) \end{vmatrix} = 0. \tag{13}$$

Similarly, as in the case b), $T(t)$ is the ratio of two determinants, and represents the interpolation function of $x(t)$ by trigonometrical polynomials.

d) Instead of the trigonometrical functions, one can have orthogonal functions such as the eigenfunctions from the quantum mechanics in the case of a discrete spectrum.

2.1. The Case of the Von Koch Curve

As an example, let us consider the von Koch curve for the second iteration (Fig. 1). Since the intervals $p_i - p_{i-1}$ $\left(i = \overline{1,16}\right)$ are equal, we shall consider the parameter *t* divided in a similar number of coordinates of equal parts. To each t_i we have an x_i and an y_i. These are the coordinates of the p_i point. The corresponding values are systematized in Table 1. These values of *t* are given exactly, the ones of x_i similarly, with the condition that the underlined decimal is infinitely repeated, while the ones of y_i are given with three decimal precision.

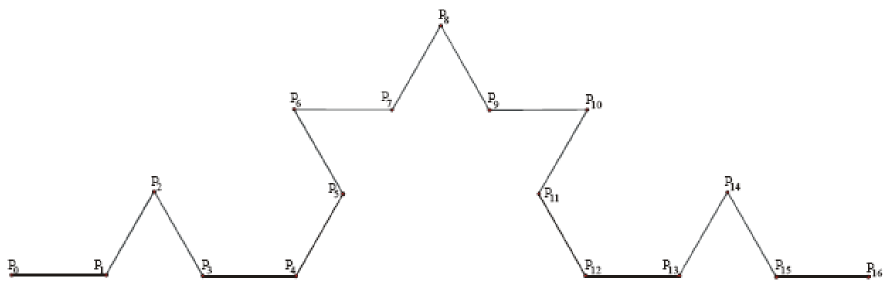

Fig. 1 – The von Koch curve for the second iteration.

Table 1

The values of coordinates $x_i(t)$ and $y_i(t)$ of the points p_i

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}
t	0	0,0625	0,125	0,1875	0,25	0,3125	0,375	0,4375	0,5	0,5625	0,625	0,6875	0,75	0,8125	0,875	0,9375	1,0
x	0	0,111	0,166	0,222	0,333	0,388	0,333	0,444	0,5	0,555	0,666	0,611	0,666	0,777	0,833	0,888	1,0
y	0	0	0,096	0	0	0,086	0,192	0,192	0,289	0,192	0,193	0,096	0	0	0,096	0	0

We shall exemplify the above mentioned facts for the case of the von Koch curve, approximated by segments (Fig. 2) and we shall refer to the first 5 segments defined by the first 6 values of *t*, i.e. for

$$t_0 = 0; \quad t_1 = \alpha; \quad t_2 = 2\alpha; \quad t_3 = 3\alpha; \quad t_4 = 4\alpha, \quad \text{with} \quad \alpha = (0.25)^2 .$$

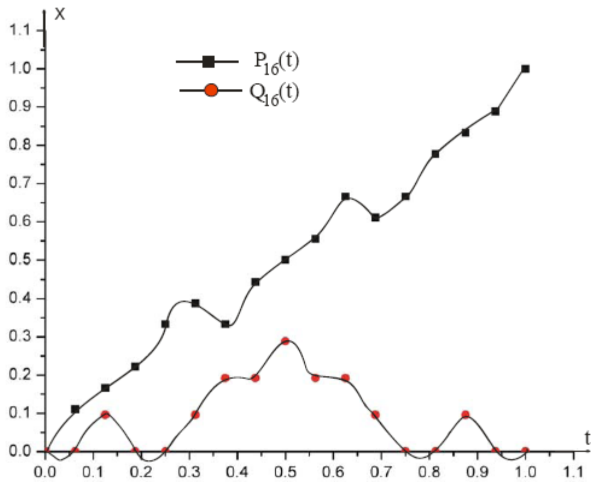

Fig. 2 – The segmented curves for the interpolation of $x(t)$ and $y(t)$.

Thus, we obtain (Fig. 3):

$$x(t)=\begin{cases}\dfrac{0.111}{\alpha}t & \ldots \quad \ldots \quad 0<t<\alpha\\ 0.111+\dfrac{0.055}{\alpha}(t-\alpha) & \ldots \quad \alpha<t<2\alpha\\ 0.166+\dfrac{0.056}{\alpha}(t-2\alpha) & \ldots \quad 2\alpha<t<3\alpha\\ 0.222+\dfrac{0.111}{\alpha}(t-3\alpha) & \ldots \quad 3\alpha<t<4\alpha\\ 0.333+\dfrac{0.055}{\alpha}(t-4\alpha) & \ldots \quad 4\alpha<t<5\alpha,\end{cases}$$

and

$$y(t)=\begin{cases}0 & \ldots \quad \ldots \quad 0<t<\alpha\\ \dfrac{0.096}{\alpha}(t-\alpha) & \ldots \quad \ldots \quad \alpha<t<2\alpha\\ 0.096-\dfrac{0.096}{\alpha}(t-2\alpha) & \ldots \quad 2\alpha<t<3\alpha\\ 0 & \ldots \quad \ldots \quad 3\alpha<t<4\alpha\\ \dfrac{0.096}{\alpha}(t-4\alpha) & \ldots \quad \ldots \quad 4\alpha<t<5\alpha.\end{cases}$$

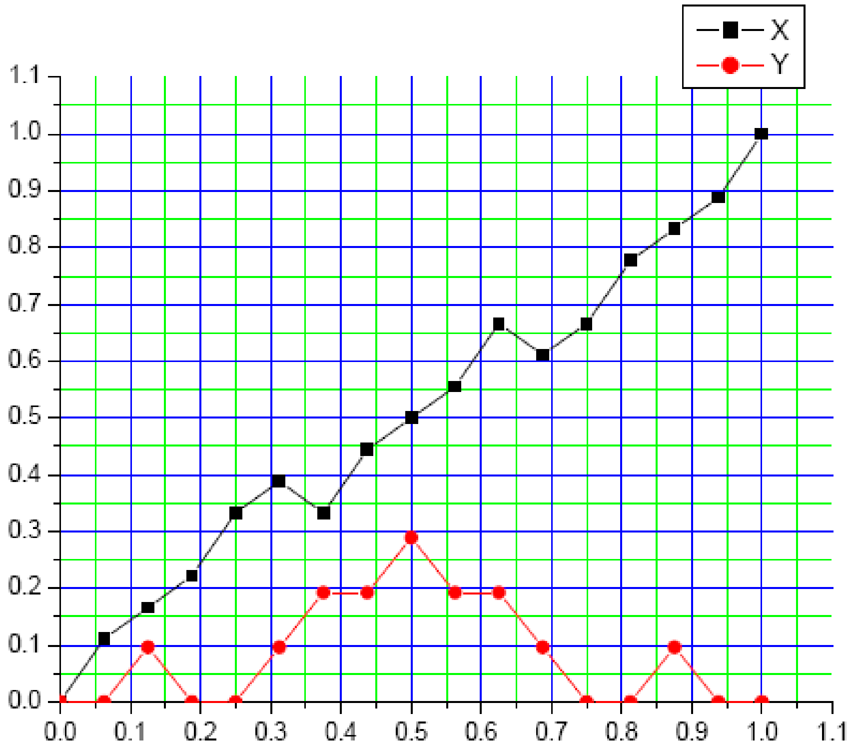

Fig. 3 – The approximation interpolation of $x(t)$ and $y(t)$ by the C^{∞} class functions (polymomials, trigonometrical polynomials, etc.).

From here, by integration, we obtain (Fig. 4):

$$\int_0^t x(t)\,\mathrm{d}t = \begin{cases} \dfrac{0.111}{2\alpha}t^2 & 0<t<\alpha \\ \dfrac{0.111}{2}\alpha + 0.111(t-\alpha) + \dfrac{0.055}{2\alpha}(t-\alpha)^2 & \alpha<t<2\alpha \\ \dfrac{0.388}{2}\alpha + 0.166(t-2\alpha) + \dfrac{0.056}{2\alpha}(t-2\alpha)^2 & 2\alpha<t<3\alpha \\ \dfrac{0.776}{2}\alpha + 0.222(t-3\alpha) + \dfrac{0.111}{2\alpha}(t-3\alpha)^2 & 3\alpha<t<4\alpha \\ \dfrac{0.776}{2}\alpha + 0.333(t-4\alpha) + \dfrac{0.055}{2\alpha}(t-4\alpha)^2 & 4\alpha<t<5\alpha, \end{cases}$$

and

$$\int_0^t y(t)\,\mathrm{d}t = \begin{cases} 0 & 0 < t < \alpha \\ \dfrac{0.096}{2\alpha}(t-\alpha)^2 & \alpha < t < 2\alpha \\ 0.096(t-2\alpha) - \dfrac{0.096}{2\alpha}(t-2\alpha)^2 & 2\alpha < t < 3\alpha \\ \dfrac{0.096}{2}\alpha & 3\alpha < t < 4\alpha \\ \dfrac{0.096}{2}\alpha + \dfrac{0.096}{2\alpha}(t-4\alpha)^2 & 4\alpha < t < 5\alpha. \end{cases}$$

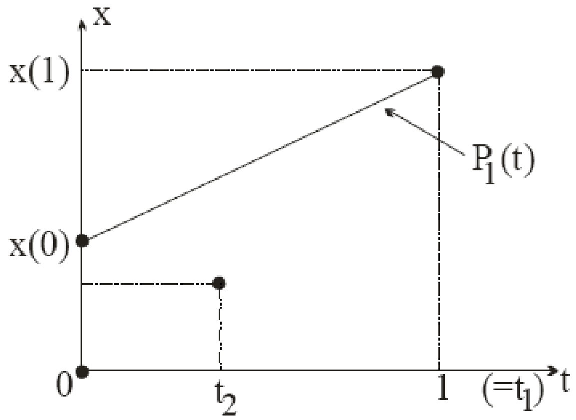

Fig. 4 – Interpolating polynomial of the first order.

The other mentioned approximations lead to similar results; they are more laborious, but simpler to be programmed on computer.

2.2. The correspondence with Cantorian space-time

As it was also shown in paragraph 1.2, we summarize at $x(t)$. This function can be approximated by polynomials. We start by the interpolating polynomial of the first order. For $t = 0$ and $t = 1$ we have

$$P_1(0) = x(0) \quad \text{and} \quad P_1(1) = x(1) \tag{14}$$

$P_1(t)$ being a line (a line segment) – Fig. 4.

Now, we add an intermediary point (it must not be in the middle!) t_2, in which the function takes the value $x(t_2)$. The polynomial of the second order (a parabola) will pass through the three points,

$$P_2(0) = x(0); \quad P_2(t_2) = x(t_2); \; P_2(1) = x(t_1). \tag{15}$$

Then, we add another point, keeping the previous ones, thus obtaining polynomials with an extra degree. Thus we find $P_n(t)$ for which

$$P_n(0) = x(0); \quad P_n(t_1) = x(t_1); \; P_n(t_2) = x(t_2); \; \dots ; \quad P_n(t_{n-1}) = x(t_{n-1}) \tag{16}$$

having the form

$$P_n = a_0 t^n + a_1 t^{n-1} + \dots + a_n . \tag{17}$$

Theorem. *We assert that the sequence* $\{P_n\}$ *is convergent and*

$$\lim_{n\to\infty} P_n(t) = x(t) \tag{18}$$

if we ascertain the fact that the values sequence, all different, for t should be compact on the entire interval (0, 1). (*The set B is compact in the set A if the closure of B is equal with A*).

Actually, let us consider $t = t_k$ and the sequential values of these interpolating polynomials in this point,

$$P_1(t_k), \dots, P_k(t_k), P_{k+1}(t_{k,\dots}). \tag{19}$$

From the way these polynomials are built, it results

$$P_k(t_k) = P_{k+1}(t_k) = \dots = P_n(t_k) \quad \text{with} \quad n \geq k . \tag{20}$$

Therefore, this will also be the limit of the sequence in $t = t_k$.

If we ascertained the fact that the sequence $\{t_k\}$ is compact in the interval $[0, 1]$, then for the values $t \notin \{t_k\}$ we find an approximation as good as possible for t.

One of the Lebesque theorems [2-5] asserts that a continuous curve, which is not differentiable in any point, has an infinite length. Since our curve is almost all over fractal self-similar, it satisfies the condition of the Lebesque theorems. Between the t' and t'' values, we can approximate the length of the fractal curve with

$$\int_{t'}^{t''} \sqrt{(\mathrm{d}P_n)^2 + (\mathrm{d}Q_n)^2} = \int_{t'}^{t''} \sqrt{\left(\frac{\mathrm{d}P_n}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}Q_n}{\mathrm{d}t}\right)^2}\, \mathrm{d}t. \tag{21}$$

When $n \to \infty$, the limit of these integrals has to be infinite. Thus, either $\mathrm{d}P_n/\mathrm{d}t$ or $\mathrm{d}Q_n/\mathrm{d}t$ (both, according to our opinion) should diverge. Consequently, either the curve $x(t)$ or $y(t)$ are fractal (or both!).

However, let us consider for the moment that n is no matter how great but finite and, by an adequate normalization, let us suppose

$$\left(\frac{\mathrm{d}P_n}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}Q_n}{\mathrm{d}t}\right)^2 = X^2 + Y^2 = 1. \tag{22}$$

Then, the equation (22) may be interpreted as absolute in the X and Y coordinates and we obtain the metric [13]

$$\mathrm{d}s^2 = -\frac{\mathrm{d}h\mathrm{d}\bar{h}}{\left(h-\bar{h}\right)^2}, \tag{23}$$

for

$$h = \frac{Y \pm \mathrm{i}\sqrt{1-X^2-Y^2}}{1-X}. \tag{24}$$

Since the metric (23) is invariant to the SL(2R) group, the restriction to the quantum space-time implies the subgroup SL(2Z) that is the transformations [13]:

$$h' = \frac{ah+b}{ch+d}, \qquad \bar{h}' = \frac{a\bar{h}+b}{c\bar{h}+d}, \qquad a,b,c,d \in Z \qquad ad-bc=\pm 1 \quad \text{(25a-d)}$$

Now, let us consider a transformation element of the matrix form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Under special conditions, $c=1$, $d=0$, we can calculate the eigenvalues by means of the irrational roots [13]

$$x_1^{(n)} = (-\phi)^n, \qquad x_2^{(n)} = (1/\phi)^n,$$

where $\phi = \left(\sqrt{5}-1\right)/2$ is the golden mean of Cantorian (El Naschie's) $\varepsilon^{(\infty)}$ space-time theory [14-21]. This clearly shows the connection between the geometry induced by the previous transformation, KAM theorem, the VAK and El Naschie's $\varepsilon^{(\infty)}$ theory [15-21].

3. CONCLUSIONS

The aim of the present paper was to establish a framework of using the integro-differential equations in the fractal theory. The main conclusions are as follows: i) using an interpolation method, the issue of primitive - indefinite integral in the theory of fractal space-time is established; ii) some examples of interpolation are given (interpolation by segments, Lagrange interpolation polynomials, interpolation by the Fourier's series); iii) the von Koch curve case is analyzed; iv) a correspondence with El Naschie's $\varepsilon^{(\infty)}$ space-time is determined.

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