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PERFECT FLUID BIANCHI TYPE I COSMOLOGICAL MODELS WITH TIME VARYING G AND Λ

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Abstract. Bianchi type I cosmological models containing perfect fluid with time varying G. and Λ have been presented. The solutions obtained represent an expansion scalar θ bearing a constant ratio to the anisotropy in the direction of space like unit vector λ^i. Of the two models obtained, one has negative vacuum energy density which decays numerically. In this model, we obtain $\Lambda \sim H^2$, $\Lambda \sim \frac{R_{44}}{R}$ and $\Lambda \sim T^{-2}$ (T is cosmic time) which is in accordance with main dynamical laws for the decay of Λ. The deceleration parameter for this model is $q = 2$. The second model reduces to a static solution with repulsive gravity.

Key words: Bianchi type I universe; varying G and Λ; cosmology theory.

1. INTRODUCTION

In order to explain the huge difference between the small cosmological constant observed today and the vacuum energy density expected from the quantum field theory calculations, several proposals have been put forward in the last few years [1]. The simplest way out of this problem is to consider a varying cosmological term, which decays from huge value at initial times to the small value observed nowadays in an expanding universe [2-4]. Several phenomenological models have been suggested by considering Λ as a function of time [3-10]. On the basis of large number hypothesis, variation of Newton's gravitational parameter was suggested by Dirac [11]. As G couples geometry to matter, it is reasonable to consider $G = G(t)$ in an evolving universe when one considers $\Lambda = \Lambda(t)$. Models with time dependent cosmological term Λ and gravitational parameter G have received considerable attention in recent years for various reasons [12-16].

In addition to cosmological term Λ, there is another quantity, the anisotropy of the universe which is supposed to play a fundamental role in the evolution and dynamics of the early universe. It is believed that the early universe was characterized by a highly irregular expansion mechanism which isotropised later

[17] in the course of their expansion. The anisotropy left out by the era of decoupling is only of order about 10^{-5} as revealed CMB observations. In a recent paper, Vishwakarma [18] proposed that whatever physical processes are responsible for the evolution of one parameter should also be responsible for the evolution of others, implying that different parameters are coupled together somehow. We assume a relation between volume expansion θ and shear σ_{ij}. This assumption is similar to the condition that σ/θ is constant, as proposed by Collins et al. [19]. In a recent paper, Kilinc [20] has considered Bianchi type I universe with the condition that volume expansion θ is proportional to the eigen value σ_{11} of shear tensor σ_{ij}. We generalize this condition by taking the volume expansion θ to have a constant ratio to the anisotropy in the direction of unit space – like vector λ^i [21].

In this paper, we investigate Bianchi type I space-time containing perfect fluid with time varying G, Λ and the volume expansion θ bearing a constant ratio to the anisotropy. Cosmological consequences of the models are also discussed.

2. FIELD EQUATIONS AND SOLUTIONS

The line element for Bianchi type I space-times is taken in the orthogonal form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\mathrm{d}x^2 + B^2(t)\mathrm{d}y^2 + C^2(t)\mathrm{d}z^2. \tag{1}$$

The distribution of matter in the space-time consists of perfect fluid given by the energy-momentum tensor

$$T_{ij} = (\rho + p)v_i v_j + p g_{ij} \tag{2}$$

satisfying the equation of state

$$p = \omega\rho, \qquad 0 \le \omega \ (\text{constant}) \le 1. \tag{3}$$

Here, ρ, p and v_i are respectively the energy density, pressure and unit flow vector of the fluid satisfying $v_i v^i = -1$.

We consider G and Λ as functions of cosmic time t. In comoving coordinates Einstein field equations

$$R_{ij} - \frac{1}{2}R g_{ij} = -8\pi G(t) T_{ij} + \Lambda(t) g_{ij}, \tag{4}$$

for the metric (1), with perfect fluid give rise to

$$8\pi G p - \Lambda = -\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4 C_4}{BC}, \tag{5}$$

$$8\pi Gp - \Lambda = -\frac{A_{44}}{A} - \frac{C_{44}}{C} - \frac{A_4 C_4}{AC}, \tag{6}$$

$$8\pi Gp - \Lambda = -\frac{A_{44}}{A} - \frac{B_{44}}{B} - \frac{A_4 B_4}{AB}, \tag{7}$$

$$8\pi G\rho + \Lambda = -\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{B_4 C_4}{BC}. \tag{8}$$

Suffix '4' on A, B, C and afterwards on ρ, Λ, G, R, H, θ, σ stands for ordinary differentiation with respect to t. Eliminating p and Λ from equations (5), (6) and (7), we obtain

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} + \frac{C_4}{C}\left(\frac{A_4}{A} - \frac{B_4}{B}\right) = 0, \tag{9}$$

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) = 0. \tag{10}$$

First integral of (9) and (10) are

$$\frac{A_4}{A} - \frac{B_4}{B} = \frac{k_1}{ABC} \tag{11}$$

and

$$\frac{B_4}{B} - \frac{C_4}{C} = \frac{k_2}{ABC}, \tag{12}$$

where k_1, k_2 are constants of integration.

An average scale factor $R(t)$ is defined by

$$R^3 = ABC. \tag{13}$$

The Hubble parameter H is given by $H = \frac{R_4}{R}$. Volume expansion θ, deceleration parameter q and shear σ for the metric (1) can be written as

$$\theta = 3H = 3\frac{R_4}{R}, \tag{14}$$

$$\sigma^2 = \frac{k_1^2 + k_2^2 + k_1 k_2}{3R^6}, \tag{15}$$

$$q = -\frac{R_{44}}{RH^2}. \tag{16}$$

Equation (15) implies that

$$\frac{\sigma_4}{\sigma} = -3H. \tag{17}$$

Equation (5) – (8) can be recast in terms of H, σ and q as

$$8\pi Gp - \Lambda = H^2\left(2q-1\right) - \sigma^2, \tag{18}$$

$$8\pi G\rho + \Lambda = 3H^2 - \sigma^2. \tag{19}$$

From (19), we obtain

$$\frac{3\sigma^2}{\theta^2} = 1 - \frac{24\pi G\rho}{\theta^2} - \frac{3\Lambda}{\theta^2}, \tag{20}$$

implying that for $\Lambda \geq 0$

$$0 < \frac{\sigma^2}{\theta^2} < \frac{1}{3}, \qquad 0 < \frac{8\pi G\rho}{\theta^2} < \frac{1}{3}.$$

The presence of a positive Λ lowers the upper limit of anisotropy whereas a negative gives more room for anisotropy.

Equation (20) can also be written as

$$\frac{\sigma^2}{3H^2} = 1 - \frac{8\pi G\rho}{3H^2} - \frac{\Lambda}{3H^2} = 1 - \frac{\rho}{\rho_c} - \frac{\rho_v}{\rho_c}, \tag{21}$$

where $\rho_c = \frac{3H^2}{8\pi G}$ is the critical density and $\rho_v = \frac{\Lambda}{8\pi G}$ is the vacuum density.

From equations (18) and (19), we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -12\pi Gp - \frac{\theta^2}{2} + \frac{3\Lambda}{2} = -12\pi G\left(\rho + p\right) - 3\sigma^2, \tag{22}$$

showing that the rate of volume expansion decreases during time evolution and the presence of a positive Λ slows down the rate of this decrease whereas a negative would promote it. From equations (18) and (19), we get

$$\Lambda = H^2\left(2-q\right) - 4\pi\left(1-\omega\right)Gp, \tag{23}$$

implying that $\Lambda \leq 0$ for $q \geq 2$.

For $p = 0$, equations (18) and (19) give

$$\frac{\Lambda}{H^2} = 1 - 2q + \frac{\sigma^2}{H^2} \tag{24}$$

$$\Omega = 1 - \frac{\Lambda}{3H^2} - \frac{\sigma^2}{3H^2} \tag{25}$$

$$\Omega = \frac{2}{3}\left(q + 1 - \frac{\sigma^2}{H^2} \right) \tag{26}$$

where $\Omega = \frac{\rho}{\rho_c}$ is the density parameter.

From equations (24) and (25) one can deduce the present values of cosmological term Λ and density parameter Ω. We observe that in an anisotropic background, value of density parameter Ω is smaller in comparison to its value in isotropic background.

In view of covariant divergence of left hand side of (4), we obtain

$$8\pi G \left\{ \rho_4 + 3(\rho + p)\frac{R_4}{R} \right\} + 8\pi\rho G_4 + \Lambda_4 = 0. \tag{27}$$

From equation (27), we observe that Λ is a constant in the absence of matter ($T_{ij} = 0$) implying that matter is essential for a time varying Λ. In the field equation (4), Λ accounts for vacuum energy with its energy density ρ_v and isotropic pressure p_v satisfying the equation of state

$$p_v = -\rho_v = -\frac{\Lambda}{8\pi G}.$$

The usual conservation law for energy-momentum tensor

$$T^j_{i;j} = 0$$

leads to

$$\rho_4 + 3(\omega + 1)\rho\frac{R_4}{R} - 0, \tag{28}$$

leaving G and Λ as some kind of coupled fields

$$8\pi\rho G_4 + \Lambda_4 = 0, \tag{29}$$

implying that Λ is a constant, whenever G is constant. Equation (28) on integration gives

$$\rho = \frac{k}{R^{3(\omega+1)}}, \quad k = \text{constant} > 0. \tag{30}$$

The system of equations (3), (5) – (8) and (29) supply only six equations in seven unknowns. To have a determinate solution we require one more condition. For this purpose, we take the volume expansion θ to have a constant ratio to the anisotropy in the direction of unit space-like vector λ^i i.e. $\frac{\theta}{\sigma_{ij}\lambda^i\lambda^j}$ is a constant [21]. Here we take, $\lambda_i = (a_1 A, a_2 B, a_3 C, 0)$, where a_1, a_2, a_3 are constants, satisfying $a_1^2 + a_2^2 + a_3^2 = 1$. In general the above condition gives rise to

$$A = B^m C^n, \tag{31}$$

where m, n are constants depending on a_1, a_2, a_3. For $m = n = 1$, it reduces to the condition considered by Kilinc [20].

Using equation (31) in equations (11) and (12), we obtain

$$C\begin{cases} = b_1\left(k_3 t + k_4\right)\dfrac{k_1 - (m-1)k_2}{(m+n+2)k_1 - (m-2n-1)k_2}, & \text{for } \dfrac{k_1}{k_2} \neq \dfrac{m-2n-1}{m+n+2} \\ = k_5 \mathrm{e}^{\frac{-k_2(m+1)}{(m+n+2)}t}, & \text{for } \dfrac{k_1}{k_2} = \dfrac{m-2n-1}{m+n+2} \end{cases}$$

and

$$B\begin{cases} = b_2\left(k_3 t + k_4\right)\dfrac{k_1 + k_2 n}{(m+n+2)k_1 - (m-2n-1)k_2}, & \text{for } \dfrac{k_1}{k_2} \neq \dfrac{m-2n-1}{m+n+2} \\ = b_5 \mathrm{e}^{\frac{-k_2(n+1)}{(m+n+2)}t}, & \text{for } \dfrac{k_1}{k_2} = \dfrac{m-2n-1}{m+n+2} \end{cases}$$

provided $m + n =. 1$. In the above k_3, k_4, k_5, b_1, b_2, b_3 are constants of integration.

For these solutions, metric (1) assumes the following forms after suitable transformations:

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}T^2 + T^{\frac{2(m+n)k_1 + 2nk_2}{(m+n+2)k_1 - (m-2n-1)k_2}}\mathrm{d}X^2 + T^{\frac{2k_1 + 2k_2 n}{(m+n+2)k_1 - (m-2n-1)k_2}}\mathrm{d}Y^2 + \\ + T^{\frac{2k_1 - 2(m-1)k_2}{(m+n+2)k_1 - (m-2n-1)k_2}}\mathrm{d}Z^2, \quad \text{for } \frac{k_1}{k_2} \neq \frac{m-2n-1}{m+n+2} \end{aligned} \tag{32}$$

$$ds^2 = -dT^2 + \exp\left\{\frac{2k_2(m-n)t}{m+n+2}\right\}dX^2 + \exp\left\{\frac{2k_2(n+1)t}{m+n+2}\right\}dY^2 + \\ + \exp\left\{\frac{-2k_2(m+1)t}{m+n+2}\right\}dZ^2, \quad \text{for} \quad \frac{k_1}{k_2} = \frac{m-2n-1}{m+n+2} \tag{33}$$

3. DISCUSSION

Average scale factor R for the model (32) is given by

$$R = T^{1/3}.$$

Hubble parameter H, volume expansion θ, shear σ and deceleration parameter q for this model are

$$\theta = 3H = \frac{1}{T}, \quad \sigma^2 = \frac{a^2}{3T^2},$$

where $a^2 = k_1^2 + k_2^2 + k_1 k_2$, and $q = 2$.

Matter density ρ, gravitational parameter G, cosmological term Λ, vacuum energy density ρ_v and critical energy ρ_c are given by:

$$\rho = kT^{-(\omega+1)}, \quad G = \frac{(1-a^2)}{12\pi k(\omega+1)}T^{\omega-1}, \quad \Lambda = \frac{(1-a^2)(\omega-1)}{3(\omega+1)}T^{-2},$$

$$\rho_v = \frac{k(\omega-1)}{2}T^{-(\omega+1)}, \quad \rho_c = \frac{k(\omega+1)}{2(1-a^2)}T^{-(\omega+1)}.$$

The model has singularity at $T = 0$. The cosmic scenario starts from a big bang at $T = 0$ and continues till $T = \infty$. Since $\frac{\sigma}{\theta} = \frac{a}{\sqrt{3}}$, the anisotropy does not die out asymptotically. The model has a constant deceleration parameter $q = 2$ and it admits a negative Λ unless ω = 1. It is to mention here that Kilinc [20] calculated incorrect expressions for Λ and G. Universe models with negative Λ are also admissible [18, 22]. For ω = 1, the model reduces to Λ = 0, G = constant and $\rho \sim T^{-2}$. At $T = 0$, ρ, G, |Λ|, σ^2 and θ are all infinite and all of them tend to zero as $T \to \infty$.

The ratio between vacuum and matter densities scales as

$$\frac{\Lambda}{8\pi G\rho} = \frac{\omega - 1}{2}.$$

The density parameter

$$\Omega = \frac{8\pi G\rho}{3H^2} = \frac{2\left(1-a^2\right)}{\omega+1} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda}{3H^2} = \frac{\left(1-a^2\right)(\omega-1)}{\omega+1}.$$

In the model, we obtain $\Lambda \sim H^2$, $\Lambda \sim \frac{R_{44}}{R}$ and $\Lambda \sim T^{-2}$ which is in accordance with the main dynamical laws one finds in literature proposed for the decay of Λ. The dynamical law $\Lambda \sim H^2$ has been proposed by Carvalho et al. [6] and considered by Salim and Waga [23], Arbab and Abdel-Rahman [24], Wetterich [25] and Arbab [7]. In view of present estimates Λ is of order of H_0^2 [13]. The dynamical law $\Lambda \sim T^{-2}$ has been considered by several authors e.g. Bertolami [2], Berman and Som [26], Berman [27], Beesham [8] to mention a few. The decay law for Λ of the form $\Lambda \sim \frac{R_{44}}{R}$ has been considered by Arbab [7]. For G to be positive, $a^2 < 1$. We observe that |Λ| decays faster than G whereas ρ, ρ_v and ρ_c scales as $T^{-(\omega+1)}$. In the model, we see that the quantity $G\rho$ satisfies the condition for a Machian cosmological solution i.e. $G\rho \sim H^2$ [28].

For the radiation epoch, we have $\omega = \frac{1}{3}$. In this case:

$$\rho = kT^{-4/3}, \quad G = \frac{\left(1-a^2\right)}{16\pi k}T^{-2/3}, \quad \Lambda = \frac{-\left(1-a^2\right)}{6}T^{-2},$$

$$\rho_v = \frac{-k}{3}T^{-4/3}, \quad \rho_c = \frac{2k}{3\left(1-a^2\right)}T^{-4/3}.$$

We observe that in this phase matter density is three times the vacuum density. The density parameter Ω in this phase is

$$\Omega = \frac{3}{2}\left(1-a^2\right) < \frac{3}{2}.$$

Also $\left|\Omega_\Lambda\right| = \frac{1}{2}\left(1-a^2\right) < \frac{1}{2}$.

For the phase dominated by dust matter, we have $\omega = 0$.

In this epoch,

$$\rho = kT^{-1}, \quad G = \frac{\left(1-a^2\right)}{12\pi k}T^{-1}, \quad |\Lambda| = \frac{\left(1-a^2\right)}{3}T^{-2}, \quad |\rho_v| = \frac{k}{3}T^{-1},$$

$$\rho_c = \frac{k}{2\left(1-a^2\right)}T^{-1}, \quad \frac{|\Lambda|}{8\pi G\rho} = \frac{1}{2}, \quad \Omega = 2\left(1-a^2\right) < 2,$$

and $|\Omega_\Lambda| = \left(1-a^2\right) < 1.$

The age of the universe is given by $T_0 = \frac{1}{3}H_0^{-1}$, which is smaller than the best estimation $T_0 = H_0^{-1}$, [29].

For the model (33), average scale factor R is given by $R = 1$.

In this model

$$\theta = 3H = 0,$$

$$\sigma^2 = \frac{b^2}{3}, \quad b^2 = \frac{3k_2\left(m^2+n^2-mn+m+n+1\right)}{m+n+2} > 0,$$

$$\rho = k,$$

$$G = -\frac{b^2}{12\pi k\left(\omega+1\right)}, \quad \Lambda = \frac{b^2\left(1-\omega\right)}{3\left(1+\omega\right)}.$$

In this case, our model reduces to a static model with repulsive gravity.

4. CONCLUSION

Bianchi type I cosmological models with time varying G and Λ are obtained. The models obtained present an expansion scalar θ bearing a constant ratio to the anisotropy in the direction of space-like unit vector λ^i. One of the two models has a negative vacuum energy density which decays numerically. In this model, we obtain $\Lambda \sim H^2$, $\Lambda = \frac{R_{44}}{R}$ and $\Lambda \sim T^{-2}$ which is in accordance with main dynamical laws for the decay of Λ. The model has a constant deceleration parameter $q = 2$ and the anisotropy in the model does not die out asymptotically. We also obtain that the model satisfies the condition for a Machian cosmological solution i.e. $G\rho \sim H^2$. For stiff matter, the model reduces to the standard Bianchi type I model with $\Lambda = 0$, G = constant and $\rho \sim T^{-2}$. The second model obtained reduces to a static universe with repulsive gravity.

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REFERENCES

1. Weinberg, S., Rev. Mod. Phys., **61**, 1 (1989).
2. Bertolami, O., Nuovo Cimento, **93**, 36 (1986).
3. Ozer, M., Taha, O., Phys. Lett., **A 171**, 363 (1986); Nucl. Phys. **B 284**, 776 (1987).
4. Freese, K. et al., Nucl. Phys., **B 287**, 797 (1987).
5. Cheu, W. and Wu, Y.S., Phys. Rev., **D 41**, 695 (1990).
6. Carvalho, J.C. and Lima, J.A.S., Phys. Rev., **D 46**, 2404 (1992).
7. Arbab. A.I., Gen. Rel. Grav., **29**, 61 (1994); **30**, 9 (1998); Class Quantum Gravity, **20**, 93 (2003); Chin. J. Astron. Astrophys, **3**, 113 (2003).
8. Beesham, A., Gen. Rel. Grav., **26**, 159 (1994).
9. Vishwakarma, R.G., Class. Quantum Grav., **17**, 3833 (2000).
10. Borges, H.A. and Carneiro, S., Gen. Relativ. Grav., **37**, 1385 (2005).
11. Dirac, P.A.M., Nature, **139**, 323 (1937).
12. Vishwakarma, R.G. and Abdussattar, Phys. Rev., **D 60**, 063507 (1999).
13. Vishwakarma, R.G., Class. Quantum Grav., **18**, 1159 (2001).
14. Pradhan, A. and Pandey, Purnima, Astrophys. Space Sci., **301**, 127 (2006).
15. Arbab, A.I. and Beesham, A., Gen. Relativ. Gravit., **32**, 615 (2000).
16. Beesham, A., Ghosh, S.G. and Lombard, R.G., Gen. Relativ. Gravit., **32**, 471 (2000).
17. Misner, C.W., Ap. J., **151**, 431 (1968).
18. Vishwakarma, R.G., Gen. Relativ. Gravit., **37**, 1305 (2005).
19. Collin, C.B., Glass, E.N. and Wilkinson, D.A., Gen. Relativ. Gravit., **12**, 805, (1980).
20. Kilinc, C.B., Astrophys. Space Sci., **289**, 103 (2004).
21. Roy, S.R., Narain, S. and Singh, J.P., Aust. J. Phys., **38**, 239 (1985).
22. Saha, Bijan, Astrophys. Space Sci. Published online, 28th March, 2006.
23. Salim, L.M. and Waga, I., Class. Quantum Grav., **10**, 1767 (1993).
24. Arbab, A.I. and Abdel-Rahaman, A-M.M., Phys. Rev., **D 50**, 7725 (1994).
25. Wetterich, C., Astron. Astrophys., **301**, 321 (1995).
26. Berman, M.S. and Som, M.M., Int. J. Theor. Phys., **29**, 1411 (1990).
27. Berman, M.S., Phys. Rev., **D 43**, 1075 (1991).
28. Berman, M.S., Int. J. Theor. Phys., **29**, 571 (1990).
29. Hansen, B.M.S. *et al.*, Astrophys. J., **574**, L 155 (2002).