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THE APPLICABILITY OF THE CAVES THEOREM TO SOLID-STATE MASERS WITHOUT INVERSION

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Abstract. The development of masers in the 1950s made possible amplifiers that were much quieter than other contemporary amplifiers. An analysis of narrow-band yields a fundamental theorem (Caves theorem) for phase-insensitive linear amplifiers; it requires that such an amplifier, will add noise as large as half-quantum of zero-point fluctuations. For phase-sensitive linear amplifiers the theorem establish a lower limit on the product of the noises added to the two phases. In the last decade it was shown theoretically that solid-state masers without inversion may be obtained in multilevel spin systems in dilute paramagnetic solids at high temperature subjected to several strong microwave fields. In the present paper the authors apply the Caves theorem to the maser without inversion in order to find out the best ways in which the proposed device can work.

Key words: microwave fields, maser without inversion, quantum noise.

1. INTRODUCTION

A linear amplifier is one whose output signal is linearly correlated with the input signal where is understood that the information is given by the complex amplitudes of the relevant modes. The discovery in the 50s of the masers made possible the amplification with very law noise in comparison with other amplifiers. That fact produced a lot of intense studies in the aria of quantum limits on noise for masers, parametric amplifiers and in general for all linear amplifiers [1–3]. The set of linear amplifiers contains the amplifiers which convert the frequency (where the output signal frequency is different by the input signal frequency) as well as phase sensitive amplifiers (where the output signal depends in an essential way on the phase of the input signal). For all this devices it was possible at the beginning of the 80s to have a unitary theory regarding the quantum limits on noise [1]. The theory developed refers only for electromagnetic signals of narrow-band (bandwidth $\Delta f << \omega/2\pi$). A linear amplifier is takes an input signal and creates an output signal from the interaction of the input with the internal freedom degree of the amplifier. Each mode will be denoted by an index α and a frequency ω_α. The operators a_α^+, a_α and b_α^+, b_α characterize the population of each mode before and after interaction. We will denote by I the set of all input modes and by O the set of all output modes. The equation of evolution for a linear amplifier is:

$$b_\alpha = \sum_{\beta \in I} (M_{\alpha\beta}^{op} a_\beta + L_{\alpha\beta}^{op} a_\beta^+) + F_\alpha, \quad \alpha \in O. \tag{1}$$

The operators $M_{\alpha\beta}^{op}$ and $L_{\alpha\beta}^{op}$ depend only of the operators of internal modes of the amplifiers and therefore they commute with a_β, a_β^+ where $\beta \in I$. The operators F_α are responsible for the noise created by the amplifier. We are interested in the F_α fluctuations and not by its mean value, therefore we can consider without loosing from generality that $< F_\alpha > = 0$. It is obvious that the mean values of the operators $M_{\alpha\beta}^{op}$ and $L_{\alpha\beta}^{op}$ will determine the amplification, and the fluctuations of these operators will introduce a supplementary noise. Of course, this noise depends on the noise of the input signal. Because we are interested in the best amplifiers, regarding the noise, we will suppose that there are not fluctuations in the values of the operators $M_{\alpha\beta}^{op}$ and $L_{\alpha\beta}^{op}$ and therefore we can replace the operators by their mean values: $\mathrm{M}_{\alpha\beta}^{\mathrm{op}} = < M_{\alpha\beta}^{op} >$ and $\mathrm{L}_{\alpha\beta}^{\mathrm{op}} = < L_{\alpha\beta}^{op} >$. In this case the equation (1) becomes:

$$b_\alpha = \sum_{\beta \in I} (\mathrm{M}_{\alpha\beta} a_\beta + \mathrm{L}_{\alpha\beta} a_\beta^+) + F_\alpha, \quad \alpha \in O. \tag{2}$$

Because the input and output operators have to respect the commutation relations ($\left[a_\alpha, a_\beta\right] = 0$ and $\left[a_\alpha, a_\beta^+\right] = \delta_{\alpha\beta}$) we have the unitarity conditions:

$$0 = \sum_{\mu \in I} (\mathrm{M}_{\alpha\mu}\mathrm{L}_{\beta\mu} - \mathrm{L}_{\alpha\mu}\mathrm{M}_{\beta\mu}) + \left[F_\alpha, F_\beta\right] \text{ and}$$

$$\delta_{\alpha\beta} = \sum_{\mu \in I} (\mathrm{M}_{\alpha\mu}\mathrm{M}_{\beta\mu}^* - \mathrm{L}_{\alpha\mu}\mathrm{L}_{\beta\mu}^*) + \left[F_\alpha, F_\beta^+\right] \quad \text{for all} \quad \alpha, \beta \in O.$$

If we consider just one mode of frequency ω, with the creation and annihilation operators a^{+}, a we can write the following relation $a = x_1 + ix_2$ where the operators x_1 and x_2 are the Hermitian real and imaginary parts. They obey the following relation $[x_1, x_2] = i/2$. The operators x_1 and x_2 are the amplitudes of the mode's quadrature phases – *i.e.*, they give the amplitudes of the mode's $\cos\omega t$ and $\sin\omega t$ oscillations. For a single mode, equation (2) becomes:

$$b_o = \mathrm{M}a_I + \mathrm{L}a_I^{+} + F\,. \tag{3}$$

We have to define what is meant by phase-insensitive amplifier. The fundamental property of a phase-insensitive amplifier is that when the input signal has phase-insensitive noise, the output, both in terms of the signal and the noise, shows no phase preference the only effect of a phase shift of the input is an equivalent phase shift of the output. We can define a phase-insensitive linear amplifier as one that satisfies the following two conditions:

(i) The expression for $< b_o >$ is invariant under arbitrary phase transformations $\varphi = \varphi_I = \theta_o$ (phase preserving amplifier) or $\varphi = \varphi_I = -\theta_o$ (phase conjugating amplifier).

(ii) If the input signal has phase insensitive noise the output signal has phase insensitive noise, *i.e.*, $< b_o^2 > = < b_o >^2$ if $< a_I^2 > = < a_I >^2$.

An amplifier that fails to meet these two conditions is called phase-sensitive linear amplifier.

If we want to continue the analysis it is convenient to introduce the complex amplitudes for a_I and b_o: $a_I = x_1 + ix_2$ and $b_o = y_1 + iy_2$. The output of a phase-sensitive linear amplifier depends in an essential way on the phase of the input. The evolution equation splits into the following equations $y_1 = (\mathrm{M}+\mathrm{L})x_1 + F_1$ and $y_2 = (\mathrm{M}-\mathrm{L})x_2 + F_2$ where $F_1 \equiv \frac{1}{2}(F + F^{+})$ and $F_2 \equiv -\frac{1}{2}i(F - F^{+})$. One can now define gains for phases: $G_1 \equiv (\mathrm{M}+\mathrm{L})^2$ and $G_2 \equiv (\mathrm{M}-\mathrm{L})^2$ and a mean gain: $G \equiv \frac{1}{2}(G_1 + G_2) = |\mathrm{M}|^2 + |\mathrm{L}|^2$ all gains being measured in units of number of quanta. The gain of a phase-insensitive amplifier is independent of phases $(G = G_1 = G_2)$.

2. THE CARACTERIZATION OF NOISE AND THE CAVES THEOREM

When the equations are written in a preferred form, the uncertainties in the output quadrature phases have the simple form $(\Delta y_p)^2 = G_p(\Delta x_p)^2 + (\Delta F_p)^2_{op}$ the first term on the right being the amplified input noise and the second term the noise added by the amplifier. Only one number is needed to characterize the noise added by a phase-insensitive amplifier, because we have $(\Delta F_1)^{op} = (\Delta F_2)^{op}$. The noise added can be characterized by an added noise number $\mathrm{A} \equiv \left|\Delta F\right|_{op}^{2} / G$ which gives the added noise referred to the input and measured in units of number of quanta. For a phase sensitive amplifier we can define the noise added for both phases $\mathrm{A_p} \equiv \left|\Delta F_p\right|_{op}^{2} / G_p,\ p = 1, 2.$

The Caves theorem claims that the quantum noise number satisfies, in the case of phase-insensitive amplifier, the following relation:

$$\mathrm{A} \geq \left|1 \mp \mathrm{G}^{-1}\right| \tag{4}$$

where the upper sign is for phase preserving amplifier and the lower one for phase conjugating amplifier. In the terms of output signal we have:

$$\left|\Delta b_O\right| = G(\Delta a_x + \mathrm{A}) \geq \frac{1}{2}G + \frac{1}{2}\left|G \mp 1\right|. \tag{5}$$

For the phase-sensitive amplifier the theorem is:

$$(\mathrm{A_1 A_2})^{1/2} \geq \frac{1}{4}\left|1 \mp (G_1 G_2)^{1/2}\right| \tag{6}$$

where the upper sign (lower) is valid when $|\mathrm{M}| \geq |\mathrm{L}|$ $(|\mathrm{L}| \geq |\mathrm{M}|)$. For the phase-sensitive amplifier the theorem imply that if we have a decrease of noise in one phase we will have also an increase of noise in the other phase. One can notice that for $G_1 \cdot G_2 = 1$ (the case of parametric degenerate amplifier) the amplifier does not add noise on the two phases.

3. THE SPECTROSCOPIC BRIDGE

The spectroscopic bridge is a proposed method of phase sensitive detection, which act like a maser without population inversion [3, 4]. The theory is a semi-classical one, which use the density matrix technique. The field is supposed to be classical and in consequence coherent. The theory proof that the output signal is linear correlated with the input one. That means that the fields are linearly correlated in second quantification, and the means number of photons are linearly correlated. This can be proof choosing an input signal [5, 6]: $\left|in\right\rangle = \exp\left(-\frac{1}{2}\left|in\right|^2\right)\sum_{n=0}^{\infty}\frac{in^n}{\sqrt{n_{in}!}}\left|n_{in}\right\rangle$ where $in = \left|in\right|e^{i\delta}$ is any complex number and

$|n_{in}\rangle$ is the state in which there are n photons with wave vector $\vec{k}$ and polarization vector $\varepsilon(\vec{k})$. Similarly we can choose a output field and imposing the relation $|in\rangle = M|out\rangle$ where M is a real number. We have $|n_{in}\rangle = \frac{[a_{\bar{r}}^{+}(\vec{k})]^{n_{\bar{r}}(\vec{k})}}{\sqrt{n_{\bar{r}}(\vec{k})!}}|0\rangle$ and similarly for the output signal. From all this equations results that the fields are linearly correlated in second quantification.

If M were not real, the first consequence would be that the classical fields are not linearly correlated.

The method is based on a principle similar with those of a transistor. Let us consider a paramagnetic sample with three levels $E_n > E_\sigma > E_m$ and three quasi-monochromatic fields related by

$$\omega_{nm} \cong \omega_{n\sigma} + \omega_{\sigma m}. \tag{7}$$

It was proved that if we have a small variation of intensity *i.e.*, $d\langle p_{ij}^2\rangle << \langle p_{ij}^2\rangle$ (here p_{ij} are the matrix elements of the hamiltonian of interaction between sample an fields), we have the following power variation $dP_{ij} = 2(E_i - E_j)Nf(material)d\langle p_{ij}^2\rangle$ where N is the total number of spins and *f(material)* is a function which depends only on the sample. In consequence: $dP_{n\sigma}/\omega_{n\sigma} \cong dP_{\sigma m}/\omega_{\sigma m} \cong -dP_{nm}/\omega_{nm} \cong$ $\cong 2\hbar Nf(material)d\langle p_{n\sigma}^2\rangle$.

From theory we have [3, 4]: $\langle p_{n\sigma}^{02}\rangle = \langle p_{\sigma m}^{02}\rangle$ and therefore:

$$\bar{N}_{n\sigma}^r \hbar\omega_{n\sigma}\mu_0 / V = \bar{N}_{\sigma m}^r \hbar\omega_{\sigma m}\mu_0 / V, \tag{8}$$

where $\bar{N}_{n\sigma}^r$ means the average number of photons with frequency $\omega_{n\sigma}$ of the reference field and V is the volume sample.

If we note $\Delta\varphi = \varphi_{n\sigma} - \varphi_{n\sigma}^r$ the phase difference between the input signal which is supposed to be of small power, coherent, with the frequency $\omega_{n\sigma}$ and the intense field of frequency $\omega_{n\sigma}$, we have the following relation

$$dp_{n\sigma} = 4(E_n - E_\sigma)Nf(material)\cos\Delta\varphi g\mu_B B_{n\sigma}^0 g\mu_B B_{n\sigma}/\hbar^2$$

where g is the spectroscopic factor, μ_B the Bohr magneton, $B_{n\sigma}^0$ and $dB_{n\sigma}$ the magnetic reference field and its variation. We can write down the following formula

$$dp_{n\sigma} = 4(E_n - E_\sigma)Nf(material)\cos\Delta\varphi g^2\mu_B^2 \sqrt{\bar{N}_{n\sigma}^r \hbar\omega_{n\sigma}\mu_0}\sqrt{\bar{N}_{n\sigma}\hbar\omega_{n\sigma}\mu_0}/V \cdot \hbar^2 \tag{9}$$

We have a similar relation for $dp_{\sigma m}$. From equation (8) and (9) one can calculate that:

$$G = 16\hbar^2 \omega_{n\sigma} \omega_{\sigma m} N^2 f^2(material) / g^2(material), \tag{10}$$

where G represents the amplification when the input signal is in phase with the intense field $\omega_{n\sigma}$. In this case the amplification is maximum. When the input signal is in anti-phase with the intense field the amplification is zero. If we supposed that $\varphi_{n\sigma}^{r} = 0$ and that the axes which we have chosen like quadrature make the δ angle with the reference axes, than $G_1 = G\cos^2\delta$ and $G_2 = G\sin^2\delta$. We have the relations $N_1^{in} = N_{in}\cos^2\varphi_{n\sigma}$ and $N_2^{in} = N_{in}\sin^2\varphi_{n\sigma}$ as well as $N_1^{out} = N_{in}\cos^2\varphi_{n\sigma}G_1$ and $N_2^{out} = N_{in}\sin^2\varphi_{n\sigma}G_2$. From these relations one can conclude $tg^2\varphi_{\sigma m} = = tg^2\varphi_{n\sigma}tg^2\gamma$. In consequence if we have $\gamma = \pi/4$ the amplification is phase insensitive, and if $\gamma \neq \pi/4$ the amplification is phase sensitive. For a small γ we have the following relation: $G_1G_2 = G^2\sin^2\gamma \cdot \cos^2\gamma \cong \cong G^2\gamma^2(1-\gamma^2) \cong G^2\gamma^2$. In consequence $G_1G_2 = 1$ if $\gamma^2 = 1/G^2$; in this case the amplifier does not add noise on either phases (is like an degenerate parametric amplifier).

4. THE CALCULATION OF NOISE IN MASER WITHOUT INVERSION

For the calculation of noise one should take into account the presence of cavity. If we have a cavity which contains a radiation of frequency v and energy W we can define the intrinsic quality factor: $Q_0 = 2\pi\nu W / P_0$ where P_0 is the absorbing power by the cavity walls. We can also define the magnetic quality factor $Q_m = 2\pi\nu W / P_m$ where P_m is the absorbing power by the sample. The cavity which contains the sample has a quality factor: $1/Q = 1/Q_0 + 1/Q_m$. The noise produces in the input and output lines as well as the walls of the cavity is of thermal nature. For this is enough to know that a body at temperature T which absorbs a percent α of the incident radiation of frequency v will re-radiate a noise power in the band $\Delta\nu$ according to:

$$P_m = \alpha\, h\nu\Delta\nu \,/\, \exp\left(\frac{h\nu}{KT}\right)^{-1} = \alpha\,\varphi(T)\,\Delta\nu. \tag{11}$$

The noises sources are: (i) the losses in input and output lines; (ii) the losses in cavity walls; (iii) the noise added by sample in concordance with Caves theorem.

The losses in input and output lines can be calculated considering an absorption coefficient α and a temperature T-constants, and a elementary length dz. We have: $dP_N = -P_N\alpha dz + \alpha\varphi(T)\Delta\nu dz$. Integrating this relation: $P_{N_2} = P_{N_1}(1-\lambda) +$

$+\lambda\varphi(T)\Delta\nu$ where: $\lambda = 1 - e^{-\alpha l}$, l is the length of the line, P_{N_1} is the noise at the beginning of the line, P_{N_2} is the noise at the end of the line.

The absorption coefficient of the walls can be calculated by: $P_0 / P_{in} = Q_c(1-G)/Q_0$ where P_0 is the power absorbed by the walls, P_{in} is the power of the input signal, $G = P_{out} / P_{in}$, and P_{out} the power of the output signal.

The noise for paramagnetic sample can be treated with the Caves theorem (we will analyze the phase insensitive case): $P_{Noise} = G_{sa}P_{in} + (1/2)(G_{sa}-1)\times$ $\times[\omega\Delta\nu\mu_0 / 2V]$ where we have: P_{in} the noise of the input signal, and G_{sa} is the sample amplification. We have:

$$G = \frac{P_{out}}{P_{in}} = \frac{P_{in} - P_0 - P_m}{P_{in}} = \frac{P_{in} - P_{0(\sigma m)} - P_m}{P_{in}}$$

and therefore $G_{sa} = \dfrac{P_{in} + P_{0(\sigma m)} - P_m}{P_{in} - P_{0(n\sigma)}}$.

Unlike the maser case, in this situation we have the input signal of frequency $\omega_{n\sigma}$ and the output frequency $\omega_{\sigma m}$.

For an amplification device, we can define the noise factor:

$$F = \frac{\text{output power noise}}{(\text{input power noise}) \times \text{amplification}} .$$

For calculating the output noise power, we will suppose that the line width in the amplification process is given by the natural width of the levels of the paramagnetic sample (the same approximation as in the maser case). We have:

$$\begin{gathered} P_{noise} = \{G[\varphi(T_i)(1-\lambda) + \lambda\varphi(T_\lambda)]\cdot\Delta\nu + \frac{Q_{c(n\sigma)}}{Q_{0(n\sigma)}}(1-G)\varphi(T_c)\cdot\Delta\nu + \\ + \frac{Q_{c(\sigma m)}}{Q_{0(\sigma m)}}(1-G)\varphi(T_c)\cdot\Delta\nu + 1/2(G_{sa}-1)[\omega_{\sigma m}\Delta\nu\mu_0]/2V\}(1-\lambda) + \lambda\varphi(T_\lambda)\cdot\Delta\nu \end{gathered} \tag{12}$$

Here T_i characterizes the input noise, T_λ is the average temperature of the input line, T_c the cavity temperature. If $\dfrac{h\nu}{KT} << 1$ we have $\varphi(T) = KT$ and therefore we can calculate the noise factor: $F = \dfrac{P_{out\ noise}}{G\cdot P_{in\ noise}} = \dfrac{P_{out\ noise}}{G\cdot KT_i \cdot\Delta\nu}$.

From direct calculations one can write:

$$F = \left\{1 - \lambda + \lambda\cdot\frac{T_\lambda}{T_i} + \left[\frac{1}{G} - 1\right]\cdot\frac{T_c}{T_i}\cdot\left[\frac{Q_{c(n\sigma)}}{Q_{o(n\sigma)}} + \frac{Q_{c(\sigma m)}}{Q_{0(\sigma m)}}\right] + \frac{G_{sa}-1}{2G}\right\}(1-\lambda) + \frac{\lambda T_\lambda}{GT_i} \tag{13}$$

If $G >> 1$ we have:

$$F = \left\{ 1 - \lambda + \lambda \cdot \frac{T_\lambda}{T_i} - \frac{T_c}{T_i} \cdot \left[\frac{Q_{c(n\sigma)}}{Q_{o(n\sigma)}} + \frac{Q_{c(\sigma m)}}{Q_{0(\sigma m)}} \right] + \frac{G_{sa}}{2G} \right\} (1 - \lambda) \tag{14}$$

Using the approximation: $P_{0(n\sigma)} = 0$ (very good quality factor on frequency $\omega_{(n\sigma)}$) we have:

$$G = \frac{P_{in} - P_{0(\sigma m)} - P_m}{P_{in}} = G_{sa} - \frac{2P_{0(\sigma m)}}{P_i} \quad \text{and therefore} \quad \frac{G_{sa}}{G} = 1 + \frac{2P_{0(\sigma m)}}{GP_i} \cong 1 \tag{15}$$

5. CONCLUSIONS

From equations (14) and (15) we have:

$$F = \left\{ \frac{3}{2} - \lambda + \lambda \cdot \frac{T_\lambda}{T_i} - \frac{T_c}{T_i} \cdot \left[\frac{Q_{c(n\sigma)}}{Q_{o(n\sigma)}} + \frac{Q_{c(\sigma m)}}{Q_{0(\sigma m)}} \right] \right\} (1 - \lambda) \tag{16}$$

If we compare the previous formula with the maser case:

$$F = \left\{ 1 - \lambda + \lambda \cdot \frac{T_\lambda}{T_i} - \frac{T_c}{T_i} \cdot \frac{Q_c}{Q_0} + \frac{T_S}{T_i} \frac{Q_c}{Q_m} \right\} (1 - \lambda) \tag{17}$$

we observe that in the spectroscopic bridge we do not have spin temperature dependence (which in our situation is close by the thermodynamic temperature). This fact is a big advantage. The explanation is that without population inversion involves reduce values of the populations on the superiors levels and in consequence a low value for the spontaneous emission. In other order of ideas, the theory and experiment [7, 8] have proved that the spontaneous emission of atoms which are in electromagnetic fields is increasing by a factor of about 1.6% when the electromagnetic field has the same frequency as the spontaneous emission; and decreasing by a factor of 0.5% when we have different frequencies. In consequence we believe that because the input signal has the same frequency as one electromagnetic field and different in rapport with the other two, we will not have an increase of spontaneous emission. As we have shown, the spectroscopic bridge can work like an amplifier sensible to phase ($G_1 G_2 = 1$), case in which it does not add noise on either phase, the "price" being the attenuation of the signal on one phase. In a recent experiment the principles of the spectroscopic bridges were confirmed by experiment [9].

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