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NONHOLONOMIC RICCI FLOWS AND RUNNING COSMOLOGICAL CONSTANT: 3D TAUB-NUT METRICS

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Abstract. The common assertion that the Ricci flows of Einstein spaces with cosmological constant can be modeled by certain classes of nonholonomic frame, metric and linear connection deformations resulting in nonhomogeneous Einstein spaces is examined in the light of the role played by topological three dimensional (3D) Taub-NUT-AdS/dS spacetimes.

Key words: Ricci flows, exact solutions, Taub-NUT spaces, anholonomic frame method.

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1. INTRODUCTION. GEOMETRIC PRELIMINARIES

There is a growing interest in the geometry of Ricci flows [1–5] and its applications in high energy physics and cosmology [6–11]. This prompts one to study systematically various examples of exactly solvable models. This paper is a continuation of [12][1], extending our previous works [13] and [14], oriented to a research on three dimensional (3D) Ricci flows with nonholonomic deformations of Thurston's geometries or topological Taub-NUT-AdS/dS (anti-de Sitter/de Sitter) configurations [15, 16]. The 3D case is the simplest one when exact solutions for nonholonomic Ricci flow equations can be constructed in explicit form and related to physically important exact solutions of the Einstein equations. Such 3D metrics are characterized by corresponding nonholonomic symmetries and, in general, possess nontrivial torsion induced by nonholonomic deformations and/or from string gravity.

[1] See there detailed motivations and outlines of the anholonomic frame method.

In this paper, we shall study 3D Ricci flows of off-diagonal metrics trivially embedded into a 4D spacetime,

$$\mathbf{g} = g_{\underline{\alpha}\underline{\beta}}(u)du^{\underline{\alpha}} \otimes du^{\underline{\beta}}, \quad (1)$$

where

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix},$$

with the indices of type $\underline{\alpha}, \underline{\beta} = (i,a), (j,b)...$ running the values $i, j = 1, 2$ and $a, b, \ldots =3, 4$ (we shall omit underlying of indices for the components with respect to coordinate basis if that will not result in ambiguities) and local coordinates labeled in the form $u = (x, y) = \{u^{\alpha} = (x^i, y^a)\}$. In order to preserve a unique system of denotations together with Ref. [12], we shall consider parametrizations when $g_{11} = \epsilon = \pm 1$ and $N_1^a = 0,$ $g_{22} = g_{22}(x^2),$ $h_{ae} = diag[h_3^a(x^2, y^b), h_4^a(x^2, y^b)]$ and $N_2^a = N_2^a(x^2, y^b),$ *i.e.*, when the coordinate x^1 will not be contained into further ansatz for metric and connections. This formal convention will allow to apply directly a number of results considered in 4D gravity and related Ricci flow solutions (we shall omit proofs and details in such cases and we shall refer the reader to the corresponding works where similar cases were analyzed for 4D or 5D constructions).

We write the normalized Ricci flow equations [1–4, 6, 7] in the form

$$\frac{\partial}{\partial \tau} g_{\underline{\alpha}\underline{\beta}} = -2R_{\underline{\alpha}\underline{\beta}} + \frac{2r}{5} g_{\underline{\alpha}\underline{\beta}}, \quad (2)$$

where $R_{\underline{\alpha}\underline{\beta}}$ is the Ricci tensor of a metric $g_{\underline{\alpha}\underline{\beta}}$ and corresponding Levi-Civita connection (by definition this connection is torsionless and metric compatible) and the normalizing factor $r = \int R dV / dV$ is introduced in order to preserve the volume V. It should be emphasized that in our works [12, 14] the running parameter τ is treated as a space-time coordinate (usually being time like or extra dimension one), which is naturally in order to relate the constructions with physical models on pseudo-Riemannian spaces. In this case, it is a more difficult task to construct exact solutions. For flows on Riemannian manifolds with τ considered as an "external" parameter labeling families of metrics, the method of generating nonholonomic exact solutions simplifies substantially.

We can represent the metric (1) in effectively diagonalized (1 + 1 + 2)-distinguished form

$$\mathbf{g} = g_{\alpha}(u)\mathbf{c}^{\alpha} \otimes \mathbf{c}^{\alpha} = \varepsilon b^1 \otimes b^1 + g_2(x^2) b^2 \otimes b^2 + h_a(u) b^a \otimes b^a, \quad (3)$$

with respect the basis

$$\mathbf{c}^{\alpha} = (b^{i} = dx^{i}, b^{a} = dy^{a} + N_{2}^{a}(u)dx^{2}) \quad (4)$$

being dual to the local basis

$$\mathbf{e}_{\alpha} = \left(e_{1} = \frac{\partial}{\partial x^{1}}, e_{2} = \frac{\partial}{\partial x^{2}} - N_{2}^{b}(u)\frac{\partial}{\partial y^{b}}, e_{b} = \frac{\partial}{\partial y^{b}} \right), \quad (5)$$

where dx^1 and $\partial_1 = \partial/\partial x^1$ are not considered in the case of 3D configurations. Such metric parametrizations and frame transforms have been introduced in the geometry of nonholonomic manifolds with associated N-connection structure defined by the set $\mathbf{N} = \{N_k^b\}$ stating a nonholonomic preferred local frame on a 3D manifold $\mathbf{V}$. We shall examine Ricci flows of 3D metrics parametrized by ansatz of type (3) when

$$g_2 = g_2(x^2),\ h_a = h_a(x^2, v),\ N_2^3 = w_2(x^2, v),\ N_2^4 = n_2(x^2, v),$$

for $y^3 = v$ being the so-called "anisotropic" coordinate.

In order to consider flows of metrics related both to the Einstein and string gravity (in the last case there is a nontrivial antisymmetric torsion field), it is convenient to work with the so-called canonical distinguished connection (in brief, d-connection) $\widehat{\mathbf{D}} = \left\{ \hat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma} \right\}$ which is metric compatible but with nontrivial torsion (see formulas (54) and related discussions for formulas (53), (56)–(60) in Appendix to [12]). Imposing certain restrictions on the coefficients N_k^b, we can satisfy the conditions that the coefficients of the canonical d-connection of the Levi-Civita $\nabla = \left\{ {}_{|}\Gamma_{\beta\gamma}^{\alpha} \right\}$ are defined by the same nontrivial values $\hat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma} = {}_{|}\Gamma_{\alpha\beta}^{\gamma}$ with respect to N-adapted basis (5) and (4).

The Ricci flow equations (2) can be written for the Ricci tensor of the canonical d-connection, $R_{\alpha\beta} = [R_{22}, R_{a2}, S_{ab}]$, and metric (3), as was considered in Refs. [14, 12],

$$\frac{\partial}{\partial \tau} g_2 = -2R_{22} + 2\lambda g_2 - h_{cd}\frac{\partial}{\partial \tau}(N_2^c N_2^d), \quad (6)$$

$$\frac{\partial}{\partial \tau} h_{ab} = -2S_{ab} + 2\lambda h_{ab}, \qquad R_{\alpha\beta} = 0 \text{ and } g_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta, \quad (7)$$

where $\lambda = r/5$, $y^3 = v$ and τ can be, for instance, the time like coordinate, $\tau = t$, or any parameter or extra dimension coordinate. The equations (6) and (7) are just the nonholonomic transform of the Ricci equations (2) if such constraints are imposed that $\hat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma} = {}_{|}\Gamma_{\alpha\beta}^{\gamma}$. The aim of this work is to show how the anholonomic frame method developed in [14,12] (for Ricci flows) and in [17–21] (for off-diagonal exact solutions) can be used for constructing exact solutions of the system of Ricci flow equations (6) and (7) describing nonholnomic deformations of 3D Taub-NUT solutions.

The structure of the paper is as follows: in Section 2 we apply the anholonomic frame method in the geometry of Ricci flows of 3D off-diagonal metrics. We construct a general class of integral varieties of Ricci flow equations and define the constrains for the Levi-Civita configurations. Section 3 is devoted to explicit solution of nonholonomic 3D Ricci flows and concentrate on two examples of constructing solutions starting with primary Taub-NUT-dS metrics. We consider details on derivation of solutions and analyze flows on holonomic and anholonomic time like coordinate. The last section is devoted to conclusions and discussion.

2. GEOMETRY OF 3D OFF-DIAGONAL RICCI FLOWS

In this section, we show how following the anholonomic frame method the 3D Ricci flow equations can be integrated in a general form.

2.1. RICCI FLOW EQUATIONS FOR OFF-DIAGONAL METRIC ANSATZ

The nontrivial components of the Ricci tensor $R_{\alpha\beta}$ (see details of a similar calculus in Ref. [19]) are

$$S_3^3 = S_4^4 = \frac{1}{2h_3h_4}\left[-h_4^{**} + \frac{\left(h_4^*\right)^2}{2h_4} + \frac{h_4^*h_3^*}{2h_4}\right], \tag{8}$$

$$R_{32} = -\frac{1}{2h_5}\left(w_2\beta + \alpha_2\right), \tag{9}$$

$$R_{42} = -\frac{h_5}{2h_4}\left(n_2^{**} + \gamma n_2^*\right) \tag{10}$$

where

$$\alpha_2 = \partial_2 h_4^* - h_4^*\partial_2 \ln\sqrt{|h_3h_4|}, \quad \beta = h_4^{**} - h_4^*\left(\ln\sqrt{|h_3h_4|}\right)^*,$$
$$\gamma = 3h_4^*/2h_4 - h_3^*/h_3, \quad \text{for} \quad h_3^* \neq 0,\ h_4^* \neq 0, \tag{11}$$

defined by h_3 and h_4 as solutions of equations (7). In the above presented formulas, it was convenient to write the partial derivative on the so-called “anisotropic” coordinate v in the form $a^* = \partial a/\partial v$.

We consider a general method of constructing solutions of the Ricci flows equations related to so-called Einstein spaces with nonhomogeneously polarized cosmological constant, when

$$S^a_b = \lambda_{[v]}(x^2, v)\delta^a_b,$$

$$R_{\alpha\beta} = 0 \text{ and } g_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta,$$

with $\lambda_{[v]}$ induced by certain string gravity ansatz, or matter field contributions, see [14, 12].

The nonholonomic Ricci flows equations (6) and (7) for the Einstein spaces with nonhomogeneous cosmological constant defined by ansatz of type (3) transform into the following system of partial differential equations consisting of two subsets of equations: The first subset of equations consists of those generated by the 3D Einstein equations for the off-diagonal metric,

$$-h_4^{**} + \frac{\left(h_4^*\right)^2}{2h_4} + \frac{h_4^* h_3^*}{2h_4} = 2h_3 h_4 \lambda_{[v]}(x^2, v), \tag{12}$$

$$w_2\beta + \alpha_2 = 0, \tag{13}$$

$$n_2^{**} + \gamma n_2^* = 0. \tag{14}$$

The second subset of equations is formed just by those describing flows of the diagonal, $g_{ij} = diag[\epsilon, g_2]$ and $h_{ab} = diag[h_3, h_4]$, and off diagonal, w_2 and n_2, metric coefficients,

$$\frac{\partial}{\partial\tau} g_2 = -h_3 \frac{\partial}{\partial\tau}\left(w_2 w_2\right) - h_4 \frac{\partial}{\partial\tau}\left(n_2 n_2\right), \tag{15}$$

$$\frac{\partial}{\partial\tau} h_a = 2\lambda_{[v]}(x^2, v) h_a. \tag{16}$$

The aim of the next section is to show how we can integrate the equations (12)–(16) in a quite general form.

2.2. INTEGRAL VARIETIES FOR 3D RICCI FLOW EQUATIONS

The equation (12) relates two nontrivial v-coefficients of the metric coefficients $h_3(x^2, v)$ and $h_4(x^2, v)$ depending on three coordinates but with partial derivatives only on the third (anisotropic) coordinate. As a matter of principle, we can fix h_3 (or, inversely, h_4) to describe any physically interesting situation being, for instance, a solution of the 3D solitonic, or pp-wave equation, and then we can try to define h_4 (inversely, h_3) in order to get a solution of (12).

Here we note that it is possible to solve such equations for any $\lambda_{[v]}(x^2, v)$, in a general form, if $h_4^* \neq 0$ (for $h_4^* = 0$, there are nontrivial solutions only if $\lambda_{[v]} = 0$). Introducing the function

$$\phi(x^2, v) = \ln\left|h_4^*/\sqrt{|h_3 h_4|}\right|, \quad (17)$$

we write that equation in the form

$$\left(\sqrt{|h_3 h_4|}\right)^{-1}\left(e^{\phi}\right)^* = -2\lambda_{[v]}. \quad (18)$$

Using (17), we express $\sqrt{|h_3 h_4|}$ as a function of ϕ and h_4^* and obtain

$$|h_4^*| = -(e^{\phi})^*/4\lambda_{[v]} \quad (19)$$

which can be integrated in general form,

$$h_4 = h_{4[0]}(x^i) - \frac{1}{4}\int dv \frac{\left[e^{2\phi(x^2,v)}\right]^*}{\lambda_{[v]}(x^2, v)}, \quad (20)$$

where $h_{4[0]}(x^2)$ is the integration function. Having defined h_4 and using again (17), we can express h_3 via h_4 and ϕ,

$$|h_3| = 4e^{-2\phi(x^2,v)}\left[\left(\sqrt{|h_4|}\right)^*\right]^2. \quad (21)$$

The conclusion is that prescribing any two functions $\phi(x^2, v)$ and $\lambda_{[v]}(x^2, v)$ we can always find the corresponding metric coefficients h_3 and h_4 solving (12). Following (21), it is convenient to represent such solutions in the form

$$h_4 = \epsilon_4\left[b(x^2, v) - b_0(x^2)\right]^2$$
$$h_3 = 4\epsilon_3 e^{-2\phi(x^2,v)}\left[b^*(x^2, v)\right]^2$$

where $\epsilon_a = \pm 1$ depending on fixed signature, $b_0(x^2)$ and $\phi(x^2, v)$ can be arbitrary functions and $b(x^2, v)$ is any function when b^* is related to ϕ and $\lambda_{[v]}$ as stated by the formula (19). Finally, we note that if $\lambda_{[v]} = 0$, we can relate h_3 and h_4 solving (18) as $\left(e^{\phi}\right)^* = 0$.

For any couples h_3 and h_4 related by (12), we can compute the values α_2, β and γ (11). This allows us to define the off-diagonal metric (N-connection) coefficients w_2 solving (13) as algebraic equations,

$$w_2 = -\alpha_2/\beta = -\partial_2\phi/\phi^*. \tag{22}$$

We emphasize, that for the vacuum Einstein equations there can be solutions of (12) resulting in $\alpha_2 = \beta = 0$. In such cases, w_2 can be arbitrary functions on variables (x^2, v) with finite values for derivatives in the limits $\alpha_2, \beta \to 0$ eliminating the "ill-defined" situation $w_2 \to 0/0$. For the Ricci flow equations with nonzero values of $\lambda_{[v]}$, such difficulties do not arise. The second subset of N-connection (off-diagonal metric) coefficients n_2 can be computed by integrating two times on variable v in (14), for given values h_3 and h_4. One obtains

$$n_2 = n_{2[1]}(x^2) + n_{2[2]}(x^2)\hat{n}_2(x^2, v), \tag{23}$$

where

$$\hat{n}_2(x^2, v) = \int h_3(\sqrt{|h_4|})^{-3} dv, \quad h_4^* \neq 0;$$
$$= \int h_3 dv, h_4^* = 0;$$
$$= \int (\sqrt{|h_4|})^{-3}\, dv, \quad h_3^* = 0,$$

and $n_{2[1]}(x^2)$ and $n_{2[2]}(x^2)$ are integration functions.

We conclude that any solution (h_3, h_4) of the equation (12) with $h_4^* \neq 0$ and non-vanishing $\lambda_{[v]}$ generates the solutions (22) and (23), respectively, of equations (13) and (14). Such solutions (of the Einstein equations) are defined by the mentioned classes of integration functions and prescribed values for $b(x^2, v)$ and $\psi(x^2)$. Further restrictions on (ε, g_2) and (h_3, h_4) are necessary in order to satisfy the equations (15) and (16) relating flows of the metric and N-connection coefficients in a compatible manner. It is not possible to solve in a quite general form such equations, but in the next section we shall give certain examples of such solutions defining flows of the Taub-NUT like metrics.

2.3. EXTRACTING SOLUTIONS FOR THE LEVI-CIVITA CONNECTION

The method outlined in the previous section allows us to construct integral varieties for the Ricci flow equations (12)–(16) derived for the canonical d-connection with nontrivial torsion, see formulas (56) and (52) in Appendix to Ref. [12]. We can restrict such integral varieties (constraining the off-diagonal metric, equivalently, N-connection coefficients w_2 and n_2 and related integration functions) in order to generate solutions for the Levi-Civita connection. The conditions ${}_{|}\Gamma^{\gamma}_{\alpha\beta} = \hat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}$ (*i.e.* the coefficients of the Levi-Civita connection are equal to the coefficients of the canonical d-connection, both classes of coefficients being computed with respect to the *N*-adapted bases (4) and (5)) hold true if the equations (see (60) in Appendix to Ref. [12])[2] are satisfied:

$$\frac{\partial h_3}{\partial x^2} - w_2 h_3^* - 2w_2^* h_3 = 0, \tag{24}$$

$$\frac{\partial h_4}{\partial x^2} - w_2 h_4^* = 0, \tag{25}$$

$$n_2^* h_4 = 0. \tag{26}$$

The relations (24) and (25) are equivalent for the general solutions h_3, see (21), h_4, see (20) and w_2, see (22), generated by a function $\phi(x^2, v)$ (17) if $\phi \to \phi - \ln 2$, when

$$\phi = \ln\left|\left(\sqrt{|h_4|}\right)^*\right| - \ln\left|\left(\sqrt{|h_3|}\right)\right|$$

and

$$w_2 = (h_4^*)^{-1}\frac{\partial h_4}{\partial x^2} = -(\phi^*)^{-1}\frac{\partial \phi}{\partial x^2},$$

where $\phi = const$ is possible only for the vacuum Einstein solutions. In a particular case, we can consider any parametrization of type $w_2 = \hat{w}_2(x^2)q(v)$ for some functions $\hat{w}_k(x^i)$ and $q(v)$. The condition (26) for $h_4 \neq 0$ constrains $n_3^* = 0$ which holds true if we put the integration functions $n_{2[2]} = 0$ in (23), when $n_2 = n_{2[1]}(x^2)$.

The final conclusion in this section is that taking any solution of equations (12), (13) and (14) we can restrict the integral varieties to such integration functions satisfying the conditions when the torsionless configurations for the Levi-Civita connection are extracted.

3. NONHOLONOMIC 3D RICCI FLOWS AND TAUB-NUT-dS METRICS

We analyze the anholonomic frame method of constructing solutions defining Ricci flows and exact solutions for three dimensional (3D) spacetimes with negative cosmological constant $\lambda = -1/l^2$. The primary metrics are those for a

[2] We emphasize that the connections on (pseudo) Riemannian and/or Riemann–Cartan spaces are not defined as tensor objects. If their coefficients are equal with respect to one frame, they can be very different with respect to other frames.

$U(1)$ fibration over 2D spaces with constant curvature. By nonholonomic deformations we shall transform such spaces into 3D manifolds, or foliations (because in this case the nonholonomic structure is integrable), with effective cosmological "constant" (anisotropically depending on some coordinates, or running in time) polarized by string corrections, Ricci flows, nontrivial torsion contributions.

3.1. SOLUTIONS FOR 3D RICCI FLOWS

We deform a primary metric $\breve{\mathbf{g}} = \left[\check{g}_2, \check{h}_a, \check{N}_2^a\right]$ (in the next section such coefficients will be stated to define certain 3D Taub-NUT like metrics) by considering polarizations coefficients η_2, η_a, η_2^a resulting in the coefficients of ansatz (3),

$$g_2 = \eta_2(x^2, v)\check{g}_2, \quad h_a = \eta_a(x^2, v)\check{h}_a, \quad N_2^a = \eta_2^a(x^2, v)\check{N}_2^a. \tag{27}$$

In explicit form, such coefficients will define nonholonomic 3D Ricci flows of certain type primary metrics. In this section we shall consider details and examples on constructing nonholonomic Ricci flow solutions.

3.1.1. General solutions for the Ricci flow equations

The set of solutions of (12) is parametrized by any functions $h_3(x^2, v)$ and $h_4(x^2, v)$ related by the condition (21), *i.e.* when

$$|h_3| = 4e^{-2\phi(x^2,v)}\left[\left(\sqrt{|h_4|}\right)^*\right]^2, \tag{28}$$

for $h_4^* \neq 0$ and $\phi(x^2, v)$ is a function to be computed from

$$|h_4^*| = -(e^{\phi})^*/4\lambda_{[v]} \tag{29}$$

where $\lambda_{[v]}(x^2, v)$ is the "vertically polarized cosmological constant. If $\lambda_{[v]} \to 0$, we have to take $\phi \to 0$ such way that h_4^* does not vanish.

The N-connection coefficients w_2 and n_2 are respectively defined by formulas (22) and (23), when

$$w_2 = -\partial_2\phi/\phi^* \tag{30}$$

and

$$n_2 = n_{2[1]}(x^2) + n_{2[2]}(x^2)\hat{n}_2(x^2, v), \tag{31}$$

where

$$\hat{n}_2(x^2, v) = \int h_3(\sqrt{|h_4|})^{-3} dv, \quad h_4^* \neq 0;$$
$$= \int h_3 dv, \quad h_4^* = 0;$$
$$= \int \left(\sqrt{|h_4|}\right)^{-3} dv, \quad h_3^* = 0.$$

One should note that the coefficients (28)–(30) and (31) for the ansatz (3) were computed to define exact solutions for the 3D Einstein equations with prescribed polarizations of cosmological constants, for the canonical d-connection. We can extract 3D (pseudo) Riemannian foliations if we impose further constraints on the *N*-connection coefficients in order to extract torsionless configurations for the Levi-Civita connection. This is possible for any parametrizations of type $w_2 = \check{w}_2(x^2)q(v)$ and if the integration function $n_{2[2]}(x^2)$ is stated to be zero.

The flow equations for the 3D ansatz (3), derived from the equations (15) and (16),

$$\frac{\partial}{\partial \tau} g_2 = -2w_2 h_3 \frac{\partial}{\partial \tau} w_2 - 2n_2 h_4 \frac{\partial}{\partial \tau} n_2, \tag{32}$$

$$\frac{\partial}{\partial \tau} h_a = 2\lambda_{[v]}(x^2, v) h_a. \tag{33}$$

In explicit form, families of solutions of these equations can be generated by fixing $\tau = x^2$, or $\tau = v$, and integrating the equations for certain prescribed values $\lambda_{[v]}(x^2, v)$. In a particular case, we can state that the target solutions define spacetimes with effective cosmological constant induce from string gravity, when $\lambda_{[h]} = \lambda_{[v]} = -\frac{\lambda_{[H]}^2}{4}$ (see Appendix in Ref. [12]); we have to solve the equations (32) and (33) for a such type prescribed cosmological constant. Inversely, we can consider that $\lambda_{[v]}(x^2, v)$ defines a nonhomogeneous polarization of the cosmological constant $\lambda = -1/l^2$ in 3D gravity, defining self-consistent non trivial Ricci flows under nonholonomic transforms.

We can chose the function $\phi(x^2, v)$ from (28) and (29) to have $|h_3| = |h_4|$.[3] For such parametrization, the equations (32) and (33) simplify substantially,

$$\frac{\partial}{\partial \tau} g_2(x^2) = -h_3 \frac{\partial}{\partial \tau}\left\{\left[w_2(x^2, v)\right]^2 \pm \left[n_2(x^2, v)\right]^2\right\},$$

$$\frac{\partial}{\partial \tau} \ln\left|\sqrt{|h_3(x^2, v)|}\right| = \lambda_{[v]}(x^2, v),$$

[3] For 3D solutions, this can be achieved by a corresponding 2D coordinate transform of x^2 and v.

where we take the sign "+" if the coordinates y^3 and y^4 have the same signature, or "–" for different signatures. For $n_2 = 0$, which is always possible if we state that the integration functions in (31) are zero (in this case we generate the Levi-Civita configurations), we can define in explicit form certain classes of exact solutions derived for any prescribed values of generating function $\phi(x^2, v)$ and $\lambda_{[h]}(x^2)$. The simplest approach is to compute the effective vertical polarization of the cosmological constant, *i.e.* the function $\lambda_{[v]}(x^2, v)$. Here it should be noted that the type of solutions depends on the fact if the variable τ is holonomic or nonholonomic.

3D Ricci flows on holonomic coordinate x^2:

If the flow coordinate is taken to be the holonomic one, $\tau = x^2$, we obtain

$$
\begin{aligned}
-\frac{\partial}{\partial x^2} g_2(x^2) &= h_3(x^2, v)\frac{\partial}{\partial x^2}\left\{\left[w_2(x^2, v)\right]^2\right\},\\
\frac{\partial}{\partial x^2}\ln\left|\sqrt{|h_3(x^2, v)|}\right| &= \lambda_{[v]}(x^2, v).
\end{aligned} \tag{34}
$$

We search a class of solutions of equations (34) with separation of variables in the form

$$h_3 = h_4 = A(x^2)B(v) \tag{35}$$

when

$$\check{\phi} = \phi - \ln 2 = \ln\left|\partial_v \ln\sqrt{|h_3|}\right|$$

and $w_2 = -\partial_2 \check{\phi}/\partial_v \check{\phi}$. We obtain from the first equation that

$$|A| = A_0 + \frac{1}{8c_0}\int\sqrt{|\partial_2 g_2(x^2)|}\,dx^2 \tag{36}$$

and B must solve the equation

$$BB^{**} + c_0(B^*)^3 - B^2(B^*)^2 = 0$$

for some integration constants c_0 and A_0. For $B^* = \partial_v B \neq 0$, we write the last equation as

$$\left(\ln|B^*|\right)^* + c_0 B^*(\ln|B|)^* = (\ln|B|)^*.$$

Introducing the function $H = (\ln|B|)^*$, the equation transforms into

$$\frac{H^*}{H^2} + c_0 B^* = 0$$

which can be integrated on v, and than transformed into

$$B^{*}=\frac{B}{c_1-c_0B}$$

with the solution

$$c_1 \ln|B(v)| - c_0 B(v) = v + v_0, \quad (37)$$

for some integration constants c_0, c_1 and v_0.

Introducing $h_3 = AB$ (35), for A defined by (36) and B defined by (37), into the second equation (34), we get that such solutions can be constructed for any polarized on x^2 vertical cosmological constant

$$\lambda_{[v]}(x^2) = \partial_2 \ln\sqrt{|A|}.$$

We conclude that 3D Ricci flow solutions on holonomic variable x^2, with separation of variables can be generated for any polarized anisotropically cosmological constants with any $\lambda_{[v]}(x^2)$. As a matter of principle we can consider dependencies of type $\lambda_{[v]}(x^2, v)$ for certain configurations not admitting separation of variables.

Putting together the above formulas, we define a class of 3D metrics solving the system (34),

$$\mathbf{g} = g_2(x^2)(dx^2)^2 + A(x^2)B(v)\left[\epsilon_3(dv + w_2(x^2,v)dx^2)^2 + \epsilon_4(dy^4)^2\right] \quad (38)$$

where $g_2(x^2)$ is an arbitrary function. Certain integration constants c_0, c_1 and A_0 are considered to be defined by fixing a 3D local coordinate system and certain boundary conditions. The nontrivial N-connection coefficient is given by

$$w_2(x^2, v) = -\partial_2 \ln|A(x^2)| / \partial_v \ln|B(v)|.$$

The metric (38) describes families of Ricci flow solutions for any prescribed values of $g_2(x^2)$ and any polarizations of cosmological constant $\lambda_{[v]}(x^2) = \partial_2 \ln\sqrt{|A(x^2)|}$. The physical meaning of such solutions is that they describe Ricci flows on coordinate x^2 of metrics of type (3) with the Levi-Civita connection (when $n_2 = 0$) as flows of 3D Einstein spacetimes with effective polarizations of the cosmological constant. In a particular case, we can say that $\lambda_{[v]}(x^2)$ is defined by any oscillating function, or one dimensional solitonic waves with a time like coordinate x^2. Such flows are with self-consistent (preserved during the flow evolution) separation of variables.

3D Ricci flows on nonholonomic coordinate *v*:

For $\tau = v$, the Ricci flows are directed by dependencies on the anholonomic coordinate,

$$0 = \frac{\partial}{\partial v}\left\{\left[w_2(x^2,v)\right]^2 \pm \left[n_2(x^2,v)\right]^2\right\},$$
$$\frac{\partial}{\partial v}\ln\left|\sqrt{|h_3(x^2,v)|}\right| = \lambda_{[v]}(x^2,v), \tag{39}$$

where $\partial_v n_2 = 0$, because $n_2 = n_{2[1]}(x^2)$, for the Ricci flows of Levi-Civita configurations.

The solution with separation of variables of system (39) when $h_3 = = A_1(x^2)B(v)$ can be performed similarly as for (34) but with $|A_1| = A_0 = const$ introduced into $B(v)$ being determined by the same type of solution like (37). The second difference from the previous case is that there are admitted solutions for vertically polarized on v cosmological constants when

$$\lambda_{[v]}(v) = \partial_v \ln\sqrt{|B(v)|}.$$

This allows us to conclude that nonholonomic 3D Ricci flows with separation variables are possible for vertical polarizations of the cosmological constant, *i.e.* for any $\lambda_{[v]}(v)$. In this case, we can also consider dependencies of type $\lambda_{[v]}(x^2,v)$ for certain configurations not admitting separation of variables.

Finally, we summarize that the solutions of (39) are parametrized

$$\mathbf{g} = g_2(x^2)(dx^2)^2 + B(v)\left[\epsilon_3 dv^2 + \epsilon_4 (dy^4)^2\right], \tag{40}$$

when $|h_3| = B(v)$ (37). For this metric, $w_2(x^2,v) = 0$ and $\lambda_{[v]}(v) = \partial_v \ln\sqrt{|B(v)|}$. The Ricci flows described by the metric (40) are on nonholonomic coordinate v for any $\lambda_{[v]}(v)$. In such cases, the cosmological constant anisotropically runs on nonholonomic variable v in the vertical 2D subspace.

3.1.2. 3D Ricci flows of Taub Nut metrics with holonomic time like coordinate

We consider the primary ansatz

$$d\check{s}^2 = \epsilon_2 \check{g}_2(x^2,v)(dx^2)^2 + \epsilon_3 \check{h}_3 (dv)^2 + \epsilon_4 \check{h}_4\left[dy^4 + \check{n}_2(x^2)dx^2\right]^2. \tag{41}$$

For the data

$$x^2 = t,\quad y^3 = r,\quad y^4 = \vartheta;\quad \epsilon_2 = -1,\quad \epsilon_3 = \epsilon_4 = 1;$$
$$\check{h}_3 = l^2/4,\quad \check{h}_4 = l^2/16n^2;$$
$$\check{g}_2 = \begin{cases} (l^2/4)\sinh^2 r; \\ (l^2/4)\cosh^2 r; \\ (l^2/4)e^{2r}; \end{cases} \quad \text{and} \quad \check{n}_2 = \begin{cases} -2n\cosh r; \\ -2n\sinh r; \\ -2ne^r; \end{cases}$$

we get three classes of exact solutions of 3D Einstein field equations with negative cosmological constant $\lambda = -1/l^2$ and nut parameter n considered in Ref. [16] and corresponding to some Lorentzian versions of the so-called Thurston's geometries [15]. We analyze nonhlonomic Ricci flows of such geometries, on holonomic coordinate $x^2 = t$.

Applying a conformal map of (41)

$$d\check{s}^2 \to d\check{s}^2_{(c)} = \left(\check{g}^2\right)^{-1} d\check{s}^2$$

and then a nonholonomic transform (the conformal transform is necessary in order to define the nonholonomic deformations in a simplified form), one generates the target ansatz

$$\begin{aligned} ds^2 = {} & \epsilon_2 \eta_2(t,r)(dt)^2 + \eta_3(t,r)\frac{\check{h}_3}{\check{g}_2(r)} dr^2 + \\ & + \eta_4(t,r)\frac{\check{h}_4}{\check{g}_2(r)}\left[d\vartheta + \eta_2^4(t,r)\check{n}_2(r)dt\right]^2 \end{aligned} \tag{42}$$

which is of type (3) with the coefficients considered to be certain functions (28), (29), (30) and (31) induced by nontrivial polarizations $\eta_2(t,r)$, $\eta_3(t,r)$, $\eta_2^4(t,r)$ and $w_2(t,r)$ in order to solve the 3D equations (12)–(14) and the flow equation (34).

For a restricted class of polarizations when $\eta_2^4 = 0$, $\eta_2 = \eta_2(t)$ and

$$g_2(t) = -\eta_2(t), \quad h_3 = \eta_3(t,r)\frac{\check{h}_3}{\check{g}_2(r)} = \eta_4(t,r)\frac{\check{h}_4}{\check{g}_2(r)},$$

we can use the ansatz of type (40)

$$\mathbf{g} = g_2(t)(dt)^2 + A(t)B(r)dr^2 + (d\vartheta)^2 \tag{43}$$

where $|h_3| = A(t)B(r)$,

$$A = A_0 + \frac{1}{8c_0}\int\sqrt{|\partial_t g_2(t)|}\,dt$$

and B is defined from

$$c_1 \ln|B(r)| - c_0 B(r) = r + r_0,$$

see formula (37). The nontrivial N-connection coefficient in (43) is computed as

$$w_2(t,r) = -\partial_t \ln|A(t)| / \partial_r \ln|B(r)|.$$

The integration constants in the above formulas are denoted c_0, c_1, A_0, r_0, $g_{2(0)}$ and ψ_0.

The metric (43) describes families of Ricci flows of the 3D Taub-NUT like solutions for any prescribed values $g_2(t)$ and polarization of cosmological constant $\lambda_{[v]}(t) = \partial_t \ln\sqrt{|A(t)|}$. We can consider more particular cases when $\lambda_{[v]}(t)$ is any oscillating function, or one dimensional solitonic wave on the time like coordinate *t*. Such flows preserve the separation of variables and the vanishing torsion for the Levi-Civita connection.

3.1.3. 3D Ricci flows on nonholonomic time like coordinate

We consider another class of Ricci flows of 3D metrics. We take the primary ansatz in the form (41) but with a reparametrization of coordinates $(t \to i\chi, \vartheta \to i\vartheta, r \to it)$ and different signature and parametrization of the nontrivial metric and *N*-connection coefficients,

$$x^3 = \vartheta,\quad y^3 = v = t,\quad y^4 = \chi;\quad \epsilon_2 = -1,\quad \epsilon_3 = 1,\quad \epsilon_4 = 1;$$
$$\check{g}_2(t) = \frac{l^2}{4}\sin^2 t,\quad \check{h}_3 = \frac{l^2}{4},\quad \check{h}_4 = \frac{l^2}{16n^2},\quad \check{n}_2(t) = 2n\cos t. \tag{44}$$

The quadratic element $d\check{s}^2$ defined by the data define another class of Thorston's geometries and exact solutions of 3D Einstein equations with negative cosmological constant also considered in Ref. [16]. Transforming conformally the primary metric, $d\check{s}^2 \to d\check{s}^2_{(c)} = \left(\check{g}_2\right)^{-1} d\check{s}^2$, and then applying a nonholonomic transform, one generates the target ansatz

$$ds^2 = \epsilon_2\eta_2(\vartheta)(d\vartheta)^2 + \epsilon_3\eta_3(\vartheta, t)\frac{\check{h}_3}{\check{g}_2(t)}dt^2 +$$
$$+\epsilon_4\eta_4(\vartheta, t)\frac{\check{h}_4}{\check{g}_2(t)}\left[d\chi + \eta_2^4(\vartheta, t)\check{n}_2(t)d\vartheta\right]^2 \tag{45}$$

which is of type (3) with the coefficients considered to be certain functions (28), (29), (30) and (31) induced by nontrivial polarizations $\eta_2(\vartheta)$, $\eta_3(\vartheta, t)$, $\eta_2^4(\vartheta, t)$ and $w_2(\vartheta, t)$ in order to solve the 3D equations (12)–(14) and the flow equation (39). This ansatz also may define Ricci flows on time like coordinate t but in a very different form than (42): In this case we shall have an angular anisotropy on ϑ and locally anisotropic flows on time t (in the previous example the flow coordinate was holonomic for the equation (34)).

The method of constructing the solutions of (12)–(14) for the ansatz (45) is completely similar to that considered in the previous example. For simplicity, we shall omit details and write down the solution applying formulas (28), (29), (30) and (31) stated for the data (44) and

$$g_2 = \epsilon_2 \eta_2(\vartheta), \quad h_3 = \epsilon_3 \eta_3(\vartheta, t) \frac{\check{h}_3}{\check{g}_2(t)}, \quad h_4 = \epsilon_4 \eta_4(\vartheta, t) \frac{\check{h}_4}{\check{g}_2(t)} \tag{46}$$
$$N_2^3 = 0, \quad N_2^4 = n_2(\vartheta, t) = \eta_2^4(\vartheta, t) \check{n}_2(t),$$

when $x^2 = \vartheta$ and $y^3 = v = t$. The solutions for Ricci flows with separation of variables are of type (40) parametrized in the form

$$\mathbf{g} = g_2(\vartheta)(d\vartheta)^2 + B(t)\left[dt^2 + (d\chi)^2\right], \tag{47}$$

when $h_3 = h_4 = B(t)$ with $B(t)$ defined from

$$c_1 \ln|B(t)| - c_0 B(t) = t + t_0,$$

see formula (37). For this metric $w_2(\vartheta, t) = 0$ and $\lambda_{[v]}(t) = \partial_t \ln \sqrt{|B(t)|}$.

The Ricci flows defined by the solutions (47) are on nonholonomic time coordinate *t*. They are defined for any coefficient $g_2(\vartheta)$ and any $\lambda_{[v]}(t)$ running in time in a manner compatible with $h_3 = h_4$. For such solutions, the cosmological constant is anisotropically polarized on angular coordinate ϑ in the "horizontal" direction and runs on nonholonomic time variable t in the vertical 2D subspace. The class of metrics (47) describes Ricci flows of conformally deformed Thorston's geometries stated by polarizations (46) and data (44).

3.2. NEW CLASSES OF 3D EXACT SOLUTIONS OF EINSTEIN EQUATIONS

There is a subclass of Thorston's geometries parametrized by stationary metrics which under nonholonomic deformations transform into other classes of exact solutions defining nonholonomic (foliated) 3D spacetimes. In this case we consider only solutions of the equations (12)–(14) but do not subject the metric and *N*-connection coefficients to solve the equations (15) and (16).

We consider the primary ansatz

$$d\check{s}^2 = \epsilon_2 \check{g}^2(v)(dx^2)^2 + \epsilon_3 \check{h}_3(dv)^2 + \epsilon_4 \check{h}_4\left[dy^4 + \check{n}_2(x^2)dx^2\right]^2. \tag{48}$$

For the data

$$x^2=\vartheta,\quad y^3=r,\quad y^4=t;\quad \epsilon_2=1,\quad \epsilon_3=1,\quad \epsilon_4=-1;$$
$$\check{h}_3=l^2/4,\quad \check{h}_4=l^2/16n^2;$$
$$\check{g}_2=\begin{cases}(l^2/4)\sinh^2 r;\\ (l^2/4)\cosh^2 r;\\ (l^2/4)e^{2r};\end{cases} \text{ and } \check{n}_2=\begin{cases}-2n\cosh r;\\ -2n\sinh r;\\ -2ne^{r};\end{cases}$$

we get other three classes of exact solutions of 3D Einstein field equations with negative cosmological constant $\lambda=-1/l^2$ and nut parameter n considered in Ref. [16] and corresponding to some Lorentzian versions of the so-called Thurston's geometries [15] (we already considered different types of primary 3D exact solutions given by ansatz (41) and data (44)). Then we introduce a conformal map $d\check{s}^2\to d\check{s}^2_{(c)}=\left[\check{g}_2\right]^{-1}d\check{s}^2$ and then a nonholonomic deformation to the off-diagonal ansatz

$$ds^2=(dx^1)^2+\eta_2(\vartheta)(d\vartheta)^2+\eta_3(\vartheta,r)\frac{\check{h}_3}{\check{g}_2(r)}\left[dr+w_2(\vartheta,r)d\vartheta\right]^2-$$
$$-\eta_4(\vartheta,r)\frac{\check{h}_4}{\check{g}_2(r)}\left[dt+\eta_3^4(\vartheta,t)\check{n}_2(r)d\vartheta\right]^2$$

were we trivially embedded the 3D ansatz into a 4D (this is necessary in order to consider cosmological constants induced from string gravity). This metric is of type (3) with polarization functions (27) and N-connection coefficients parametrized in the form

$$g_1=1,\quad g_2=\eta_2(\vartheta),\quad h_3=\eta_3(\vartheta,r)\frac{\check{h}_3}{\check{g}_2(r)},\quad h_4=\eta_4(\vartheta,r)\frac{\check{h}_4}{\check{g}_2(r)},$$
$$N_2^3=w_2(\vartheta,r),\quad N_2^4=n_2(\vartheta,r)=\eta_2^4(\vartheta,r)\check{n}_2(r). \tag{49}$$

We can use formulas (28), (29), (30) and (31) in order to write down the general solution of the Einstein equations with effective cosmological constant induced from string gravity. The solutions for $\lambda_{[v]}=-\lambda_H^2/4$ are given by the coefficients of the d-metric

$$g_1=1,\quad g_2(\vartheta)=\eta_2(\vartheta),$$
$$|h_3|=\left[\partial_r\phi(\vartheta,r)\right]^2 e^{-\phi(\vartheta,r)}\frac{\check{h}_4}{\lambda_H^2\,\check{g}_2(r)},\quad h_4=\frac{\epsilon_4}{\lambda_H^2}e^{\phi(\vartheta,r)}\frac{\check{h}_4}{\check{g}_2(r)}, \tag{50}$$

where ψ_0 and $g_{2(0)}$ are integration constants and $\phi(\vartheta, r)$ is an arbitrary function, and of *N*-connection coefficients

$$w_2 = -\partial_\vartheta \phi(\vartheta, r)/\partial_r \phi(\vartheta, r)$$

and

$$n_2 = n_{2[1]}(\vartheta) + n_{2[2]}(\vartheta)\hat{n}_k(\vartheta, r), \tag{51}$$

where

$$\hat{n}_2(\vartheta, r) = \int h_3(\sqrt{|h_4|})^{-3} dr,$$

for $h_4^*,\ h_3^* \neq 0.$ We can define the polarization functions by introducing (50) and (51) into (49).

Putting together the defined coefficients, we construct this class of 3D exact solutions generated effectively in string gravity:

$$\begin{aligned} ds^2 = (dx^1)^2 + g_2(d\vartheta)^2 + \left(\partial_r\phi\right)^2 e^{-\phi}\frac{\check{h}_4}{\lambda_H^2}\left[dr - \frac{\partial_\vartheta \phi}{\partial_r \phi}d\vartheta\right]^2 - \\ -e^{\phi}\frac{\check{h}_4}{\lambda_H^2 \check{g}_2}\left[dt + \left(n_{2[1]} + n_{2[2]}\int h_3(\sqrt{|h_4|})^{-3}\,dr\right)\check{n}_2 d\vartheta\right]^2. \end{aligned} \tag{52}$$

Such solutions are induced as nonholonomic string deformations of conformally deformed Thurston's geometries stated by the primary metrics (48). They define nonholonomic fibrations over 3D spacetime trivially embedded into the 4D spacetime. This family of solutions is generated by arbitrary integration functions $\phi(\vartheta, r),\ n_{2[1]}(\vartheta)$ and $n_{2[2]}(\vartheta)$ and integration constants ψ_0 and $g_{2(0)}.$ The subclass of solutions with trivial torsion for the Levi-Civita connection (nevertheless with nontrivial string torsion) can be extracted if we impose the condition $n_{2[2]} = 0$ and chose, for instance, $\phi(\vartheta, r) = \phi_1(\vartheta)\phi_2(r)$ to induce a parametrization of type $w_2 = \widehat{w}_2(\vartheta)q(r).$ This defines certain spacetimes as 3D foliation structures. In the limit of trivial polarizations $\eta \to 1$ and $w_2 \to 0,$ with $\lambda_{[v]} = -\lambda_H^2/4 \to -1/l^2$ the metric (52) does not transform into a solution of the 3D Einstein equations with cosmological constant $\lambda = -1/l^2$ but into a conformal transform of a such solution stated above by $d\check{s}^2_{(c)}.$ This is because, as a matter of principle, the anholonomic frame method can be applied to deform primary metrics which are not exact solutions. Nevertheless, the final result is always related to certain classes of generic off-diagonal solutions.

It should be noted that for 3D curved spaces any metric can be diagonalized by corresponding coordinate transforms. This holds true for the generated classes of solutions. We can not apply any type of coordinate transform if we do not preserve a prescribed nonholonomic/ foliated structure for new classes of locally anisotropic Taub-NUT spacetimes. Finally, we note that the solutions (52) can not be deformed (following the anhlonomic frame method applied in this work) into certain solutions of the Ricci flow equations because the time like coordinate for the considered family of ansatz was chosen to be $y^4 = t$, when the primary and target metrics do not depend on this variable by definition. So, such flow solutions can not be constructed but stationary generic off-diagonal solutions of the Einstein equations are possible.

4. OUTLOOK AND DISCUSSION

We have considered here three dimensional solutions of the Ricci flow equations, in a special case, defining flows of the Thorston's geometries and corresponding Taub-NUT like metrics. These solutions were constructed following the anholonomic frame method and the geometry of 3D foliated manifolds, in general, with nontrivial torsion. The novelty of the method is that it allows to consider off-diagonal Ricci flows with possible constraints and nonholonomic deformations resulting in effectively nonholomogeneous cosmological constants with anisotropic polarizations modeling flows of the Einstein spaces. The solutions can be generalized for four dimensions as is considered in the partner paper [12].

It was found that, depending on the time like flow coordinate being holonomic or anholonomic, the types of flows and admissible polarizations of the cosmological constants are very different. In the first case both are possible, for instance, angular and time dependencies of the cosmological constant, but in the second case only configurations with running in time cosmological "constants" are self-consistent. For certain classes of Thurston's geometries it is not possible to generate anholonomic Ricci flows, following our methods of solutions. Nevertheless, generalizations to new classes of exact 3D solutions defining generic off-diagonal Einstein spaces can be obtained. Here, one should note that even in 3D every metric can be diagonalized by coordinate transforms, the off-diagonal metric terms have a special physical importance if certain nonholonomic constraints and contributions of torsion, for instance, from string gravity, are taken into consideration.

The method elaborated in this work can be applied for any signatures of metric and for various primary metrics and linear connections (even they do not define an exact solution) nonholonomically deformed in order to generate exact solutions of the Ricci flow equations. In a number of cases, there are nontrivial limits of the flow solutions to certain classes of exact solutions of the field equations.

We leave for future work the study of thermodynamic properties of such solutions which can be considered as some equilibrium states of a corresponding locally anisotropic space time kinetic model [22, 23] and can be used as test “flow” grounds for AdS/CFT correspondence and various models with nontrivial topology [24, 25].

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