

Dedicated to Prof. Dumitru Barbu Ion's 70th Anniversary

COHERENT STATES ASSOCIATED TO THE JACOBI GROUP

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Abstract. We review and discuss some properties of the coherent states associated to the Jacobi group.

Key words: coherent states, squeezed states, Jacobi group.

1. INTRODUCTION

The coherent states (CS) offer a useful connection between classical and quantum mechanics [1–5]. In several references [6, 7] we have constructed CS attached to the Jacobi group. It is well known that the Jacobi group appears in Quantum Mechanics, Geometric Quantization, Optics [8–16]. The mathematicians have given the name “Jacobi group” to the semidirect product of the Heisenberg-Weyl group and the symplectic group [17]. The same group is known to physicists under other names, as the Schrödinger group [18]. Also the name “Weyl-symplectic” group is used for the same semi-direct product of the Heisenberg-Weyl group and the symplectic group [8, 19, 20].

In [6] we have constructed generalized CS attached to the Jacobi group $G_1^J := H_1 \rtimes \text{SU}(1, 1)$, based on the homogeneous Kähler manifold $\mathcal{D}_1^J = H_1/\mathbb{R} \times \text{SU}(1, 1)/\text{U}(1) = \mathbb{C}^1 \times \mathcal{D}_1$. \mathcal{D}_n denotes the Siegel unit ball [21, 22] and H_n is the $(2n + 1)$ -dimensional real Heisenberg-Weyl group with Lie algebra \mathfrak{h}_n . When expressed in appropriate coordinates on the manifold $\mathcal{X}_1^J := \mathbb{C} \times \mathcal{H}_1$, where \mathcal{H}_1 is the Siegel upper half plane $\mathcal{H}_1 = \{v \in \mathbb{C} \mid \Im(v) > 0\}$, the Kähler two-form ω_1 derived from the Kähler potential obtained from the scalar product of Perelomov's coherent state vectors based on \mathcal{D}_1^J , is identical with the one considered by Kähler-Berndt [23–27], here denoted ω_1' . This ω_1' describes the geometry of manifolds on which

the “gaussons” [28] are constructed [7, 3]. In the present paper we also give more details about this identification. We also recall our generalization of the full construction to the “ n ” dimensional Jacobi group $G_n^J = H_n \rtimes \text{Sp}(n, \mathbb{R})$ in [7], where the associated coherent states are based on the manifold $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$. Our method of finding representations of the Lie algebra is based on our previous papers [29–31], as a particular case of realization of the so called coherent state algebras [22, 32, 33] by holomorphic differential operators with polynomial coefficients.

The paper is laid out as follows. §2 fixes the definition of the Jacobi algebra \mathfrak{g}_1^J . §3 recall the construction of CS vectors attached to the Jacobi group G_1^J defined on the manifold \mathcal{D}_1^J and also of the normalized squeezed CS vectors [34]. The main information is extracted from [6], but we also present in Proposition 1 a very suggestive formula of the series expansion for CS vectors. The action of the Jacobi group G_1^J on the base manifold \mathcal{D}_1^J can be extracted from the first part of Proposition 2 of §4, which gives the resolution of unity, sometimes [3] considered as characterization of CS. The Kähler two form ω_1 on \mathcal{D}_1^J is reproduced also in §4. The last section is devoted to Kähler-Berndt’s approach. In Remark 1 we present in full details a proof of a known result from the papers of Kähler and Berndt, the action of the real Jacobi group $G_0^J(\mathbb{R})$ on \mathcal{X}_1^J . During the presentation, I indicate the significance in Physics of the mathematical formulas.

2. THE JACOBI ALGEBRA \mathfrak{g}_1^J

The Heisenberg-Weyl group is the group with the 3-dimensional real Lie algebra

$$\mathfrak{h}_1 \equiv \mathfrak{g}_{HW} = \left\langle is1 + xa^+ - \bar{x}a \right\rangle_{s \in \mathbb{R}, x \in \mathbb{C}}, \quad (1)$$

where a^+ (a) are the boson creation (respectively, annihilation) operators which verify the CCR (4a).

Let us also consider the Lie algebra of the group $\text{SU}(1, 1)$:

$$\mathfrak{su}(1, 1) = \left\langle 2i\theta K_0 + yK_+ - \bar{y}K_- \right\rangle_{\theta \in \mathbb{R}, y \in \mathbb{C}}, \quad (2)$$

where the generators $K_{0,+,-}$ verify the standard commutation relations (4b).

The Jacobi algebra is defined as the the semi-direct sum

$$\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1, 1), \quad (3)$$

where \mathfrak{h}_1 is an ideal in \mathfrak{g}_1^J , i.e. $[\mathfrak{h}_1, \mathfrak{g}_1^J] = \mathfrak{h}_1$, determined by the commutation relations:

$$[a, a^+] = 1, \quad (4a)$$

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0, \quad (4b)$$

$$[a, K_+] = a^+, \quad [K_-, a^+] = a, \quad (4c)$$

$$[K_+, a^+] = [K_-, a] = 0, \quad (4d)$$

$$[K_0, a^+] = \frac{1}{2}a^+, \quad [K_0, a] = -\frac{1}{2}a. \quad (4e)$$

The general scheme [31] associates to elements of the Lie algebra \mathfrak{g} differential operators: $X \in \mathfrak{g} \rightarrow \mathbb{X}$. We recall the following result established in [6]:

Lemma 1. *The differential action of the generators (4a)–(4e) of the Jacobi algebra (3) is given by the formulas:*

$$a = \frac{\partial}{\partial z}; \quad a^+ = z + w \frac{\partial}{\partial z}; \quad (5a)$$

$$\mathbb{K}_- = \frac{\partial}{\partial w}; \quad \mathbb{K}_0 = k + \frac{1}{2}z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}; \quad (5b)$$

$$\mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w}, \quad (5c)$$

where $z \in \mathbb{C}$, $|w| < 1$.

In the proof of Lemma 1, the (un-normalized) Perelomov's CS vectors are used, and this construction is recalled in the next section.

3. COHERENT STATE VECTORS

Let us suppose that we know the derived representation $d\pi$ of the Lie algebra \mathfrak{g}_1^J (3) of the Jacobi group G_1^J . We associate to the generators a, a^+ of the HW-group and to the generators $K_{0,+,-}$ of the group $SU(1, 1)$ the operators a, a^+ , respectively $\mathbf{K}_{0,+,-}$, where $(a^+)^+ = a$, $\mathbf{K}_0^+ = \mathbf{K}_0$, $\mathbf{K}_{\pm}^+ = \mathbf{K}_{\mp}$. We impose to the cyclic vector e_0 to verify simultaneously the conditions

$$ae_0 = 0, \quad (6a)$$

$$\mathbf{K}_-e_0 = 0, \quad (6b)$$

$$\mathbf{K}_0 e_0 = k e_0; \quad k > 0, \quad 2k = 2, 3, \dots \quad (6c)$$

We have considered in (6c) the positive discrete series representations D_k^+ of $SU(1, 1)$ [35].

Perelomov's CS vectors associated to the group G_1^J with Lie algebra the Jacobi algebra (3), based on the manifold M ,

$$M := H_1/\mathbb{R} \times SU(1, 1)/U(1), \quad (7a)$$

$$M = \mathcal{D}_1^J := \mathbb{C} \times \mathcal{D}_1, \quad (7b)$$

are defined as

$$e_{z,w} := e^{za^+ + w\mathbf{K}_+} e_0, \quad z, w \in \mathbb{C}, \quad |w| < 1. \quad (8)$$

The displacement operator

$$D(\alpha) := \exp(\alpha a^+ - \bar{\alpha} a) = \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp(\alpha a^+) \exp(-\bar{\alpha} a), \quad (9)$$

verifies the relation

$$D(\alpha_2)D(\alpha_1) = e^{i\Im(\alpha_2 \bar{\alpha}_1)} D(\alpha_2 + \alpha_1). \quad (10)$$

Let us denote by S – the unitary squeezed operator – the D_+^k representation of the group $SU(1, 1)$ and let us introduce the notation $\underline{S}(z) = S(w)$, where w and z , $w \in \mathbb{C}$, $|w| < 1$, are related by (11c), (11d):

$$\underline{S}(z) := \exp(z\mathbf{K}_+ - \bar{z}\mathbf{K}_-), \quad z \in \mathbb{C}; \quad (11a)$$

$$S(w) = \exp(w\mathbf{K}_+) \exp(\eta\mathbf{K}_0) \exp(-\bar{w}\mathbf{K}_-); \quad (11b)$$

$$w = w(z) = \frac{z}{|z|} \tanh(|z|), \quad w \in \mathbb{C}, \quad |w| < 1; \quad (11c)$$

$$z = z(w) = \frac{w}{|w|} \operatorname{arctanh}(|w|) = \frac{w}{2|w|} \log \frac{1+|w|}{1-|w|}; \quad (11d)$$

$$\eta = \log(1 - w\bar{w}) = -2 \log(\cosh(|z|)). \quad (11e)$$

We introduce also the normalized (squeezed) CS vector [34])

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0. \quad (12)$$

We introduce the auxiliary operators:

$$\mathbf{K}_+ = \frac{1}{2}(a^+)^2 + \mathbf{K}'_+, \quad (13a)$$

$$\mathbf{K}_- = \frac{1}{2}a^2 + \mathbf{K}'_-, \quad (13b)$$

$$\mathbf{K}_0 = \frac{1}{2}(a^+a + \frac{1}{2}) + \mathbf{K}'_0, \tag{13c}$$

which have the properties

$$\mathbf{K}'_0 e_0 = 0, \tag{14a}$$

$$\mathbf{K}'_0 e_0 = k' e_0; \quad k = k' + \frac{1}{4}; \tag{14b}$$

$$[\mathbf{K}'_\sigma, a] = [\mathbf{K}'_\sigma, a^+] = 0, \quad \sigma = \pm, 0, \tag{15a}$$

$$[\mathbf{K}'_0, \mathbf{K}'_\pm] = \pm \mathbf{K}'_\pm; \quad [\mathbf{K}'_-, \mathbf{K}'_+] = 2\mathbf{K}'_0. \tag{15b}$$

We recall the orthonormal system of coherent states associated to the group $SU(1, 1)$:

$$e_{k,k+m} := a_{km}(\mathbf{K}_+)^m e_{k,k}; \quad a_{km}^2 = \frac{\Gamma(2k)}{m!\Gamma(m+2k)}, \tag{16}$$

and to the Heisenberg-Weyl group

$$|n\rangle = (n!)^{-\frac{1}{2}}(a^+)^n |0\rangle; \quad \langle n', n \rangle = \delta_{nn'}. \tag{17}$$

We write down the vector e_0 in (6) as

$$e_0 = e_0^H \otimes e_0^{K'}, \quad \text{where } |0\rangle \equiv e_0^H; \quad e_0^{K'} \equiv e_{k',k'}. \tag{18}$$

Proposition 1. *Perelomov's coherent state vector (8) can be written down as*

$$e_{z,w} = E(z, w)e_0^H \otimes e^{w\mathbf{K}'} e_0^{K'} \tag{19}$$

where the operator

$$E(z, w) = e^{za^+ + \frac{w}{2}(a^+)^2} \tag{20}$$

has the action

$$E(z, w)e_0^H = \sum \frac{1}{(n!)^{1/2}} \left(-\frac{w}{2}\right)^{n/2} H_n\left(\frac{iz}{\sqrt{2w}}\right) |n\rangle, \tag{21}$$

and H_n are the Hermite polynomials

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}. \tag{22}$$

The second term in (19) admits the standard expansion in the operators (16)

$$e^{w\mathbf{K}'_+} e_0^{K'} = \sum \frac{w^m e_{k',k'+m}}{m! a_{k'm}}. \tag{23}$$

The base of functions associated to the CS-vectors attached to the Jacobi group G_1^J , based on the manifold M (7b),

$$f_{|n\rangle; e_{k', k'+m}}(z, w) := (e_{\bar{z}, \bar{w}}, |n\rangle e_{k', k'+m}), \quad z \in \mathbb{C}, \quad |w| < 1, \quad (24)$$

consists of the functions

$$f_{|n\rangle; e_{k', k'+m}}(z, w) = f_{e_{k', k'+m}}(w) \frac{P_n(z, w)}{\sqrt{n!}}, \quad (25)$$

where

$$f_{e_{k, k+n}}(w) := (e_{z=0, \bar{w}}, e_{k, k+n}) = \sqrt{\frac{\Gamma(n+2k)}{n! \Gamma(2k)}} w^n, \quad |w| < 1. \quad (26)$$

is the base of holomorphic functions on \mathcal{D}_1 , and the holomorphic polynomials $P_n(z, w)$ have the expression

$$P_n(z, w) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{w}{2}\right)^k \frac{z^{n-2k}}{k!(n-2k)!}. \quad (27)$$

Let $K = K(\bar{z}, \bar{w}, z, w)$, where $z \in \mathbb{C}$, $w \in \mathbb{C}$, $|w| < 1$,

$$K := (e_0, e^{\bar{z}a + \bar{w}K_-} e^{za^+ + wK_+} e_0). \quad (28)$$

Then the reproducing kernel is

$$K = (1 - w\bar{w})^{-2k} \exp \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{2(1 - w\bar{w})}. \quad (29)$$

More generally, the kernel $K(z, w; \bar{z}', \bar{w}') := (e_{\bar{z}, \bar{w}}, e_{\bar{z}', \bar{w}'}) : \mathcal{D}_1^J \times \bar{\mathcal{D}}_1^J \rightarrow \mathbb{C}$ is:

$$K(z, w; \bar{z}', \bar{w}') = (1 - w\bar{w}')^{-2k} \exp \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2w}{2(1 - w\bar{w}')}. \quad (30)$$

The reproducing kernel (30) $K : M \times \bar{M} \rightarrow \mathbb{C}$ admits the series expansion in the base functions (25)

$$K(z, w; \bar{z}', \bar{w}') = \sum_{n, m} f_{|n\rangle; e_{k', k'+m}}(z, w) \bar{f}_{|m\rangle; e_{k', k'+m}}(\bar{z}', \bar{w}'). \quad (31)$$

The normalized squeezed state vector (12) and the un-normalized (Perelomov's coherent state) vector (8) are related by the relation

$$\Psi_{\alpha, w} = (1 - w\bar{w})^k \exp\left(-\frac{\bar{\alpha}}{2} z\right) e_{z, w}, \quad (32)$$

where $z = \alpha - w\bar{\alpha}$.

4. THE RESOLUTION OF UNITY

From first part of the next proposition we can see the holomorphic action of the Jacobi group

$$G_1^J := H_1 \rtimes \text{SU}(1, 1) \quad (33)$$

on the manifold \mathcal{D}_1^J (7b). For self-contentedness, we also reproduce the resolution of unity:

Proposition 2. *Let us consider the action $S(g)D(\alpha)e_{z,w}$, where $g \in \text{SU}(1, 1)$ has the form (34),*

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \text{where } |a|^2 - |b|^2 = 1. \quad (34)$$

$D(\alpha)$ is given by (9), and Perelomov's coherent state vector is defined in (8). Then we have the formula (35) and the relations (36), (37)–(39) below:

$$S(g)D(\alpha)e_{z,w} = \lambda e_{z_1, w_1}, \quad \lambda = \lambda(g, \alpha; z, w), \quad (35)$$

$$z_1 = \frac{\alpha - \bar{\alpha}w + z}{bw + \bar{a}}; \quad w_1 = g \cdot w = \frac{aw + b}{bw + \bar{a}}, \quad (36)$$

$$\lambda = (\bar{a} + \bar{b}w)^{-2k} \exp\left(\frac{z}{2}\bar{\alpha}_0 - \frac{z_1}{2}\bar{\alpha}_2\right) \exp i\Im(\alpha \cdot \bar{\alpha}_0), \quad (37)$$

$$\alpha_0 = \frac{z + \bar{z}w}{1 - w\bar{w}}, \quad (38)$$

$$\alpha_2 = (\alpha + \alpha_0)a + (\bar{a} + \bar{\alpha}_0)b. \quad (39)$$

The scalar product of functions from the space \mathcal{F}_K corresponding to the kernel defined by (30) on the manifold (7b) is:

$$(\phi, \psi) = \Lambda \int_{z \in \mathbb{C}; |w| < 1} \bar{f}_\phi(z, w) f_\psi(z, w) (1 - w\bar{w})^{2k} \exp - \frac{|z|^2}{1 - w\bar{w}} \exp - \frac{z^2 \bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})} dv,$$

where the value of the G^J -invariant measure dv is

$$dv = \frac{d\Re w d\Im w}{(1 - w\bar{w})^3} d\Re z d\Im z \quad (40)$$

and

$$\Lambda = \frac{4k - 3}{2\pi^2}. \quad (41)$$

The Kähler potential is calculated as the logarithm of the reproducing kernel (30), $f := \log K$, i.e.

$$f = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{2(1-w\bar{w})} - 2k \log(1-w\bar{w}), \quad (42)$$

and the Kähler two-form ω_1 is obtained via the formula:

$$-i \omega_1 = f_{z\bar{z}} dz \wedge d\bar{z} + f_{z\bar{w}} dz \wedge d\bar{w} - f_{\bar{z}w} d\bar{z} \wedge dw + f_{w\bar{w}} dw \wedge d\bar{w}, \quad (43)$$

$$-i \omega_1 = \frac{2k}{(1-w\bar{w})^2} dw \wedge d\bar{w} + \frac{A \wedge \bar{A}}{1-w\bar{w}}, \quad A = dz + \bar{\alpha}_0 dw, \quad \alpha_0 = \frac{z + \bar{z}w}{1-w\bar{w}}. \quad (44)$$

5. KÄHLER-BERNDT'S APPROACH

Rolf Berndt – alone or in collaboration – has studied the real Jacobi group $G^J(\mathbb{R})$ in several references, from which I mention [23, 36, 24, 37]. The Jacobi group appears (see explanation in [38]) in the context of the so called *Poincaré group* or *The New Poincaré group* – the double cover of the de Sitter group $SO_0(4, 1)$ – investigated by Erich Kähler as the 10-dimensional group G^K which invariants a hyperbolic metric [25–27]. Kähler and Berndt have investigated the Jacobi group $G_0^J(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^2$ acting on the manifold $\mathfrak{X}_1^J := \mathcal{H}_1 \times \mathbb{C}$, where \mathcal{H}_1 is the upper half plane $\mathcal{H}_1 := \{v \in \mathbb{C} \mid \Im(v) > 0\}$.

In Remarks 1 and 2 below, we proof two results from [24], which we need in order two express the two-form ω_1 in the coordinates used by Kähler and Berndt. The main ingredient in the proof of Remark 1 below is the Iwasawa decomposition. Let us mention that Iwasawa decomposition was largely used in applications in Optics, see e.g. [39, 15]. Note also that the linear fractional transformation expressed by the first relation (45) is the famous “abcd law” in Optics [40, 28].

Remark 1. The action of $G_0^J(\mathbb{R})$ on \mathfrak{X}_1^J is given by $(g, (v, z)) \rightarrow (v_1, z_1)$, $g = (M, l)$, where

$$v_1 = \frac{av + b}{cv + d}, \quad z_1 = \frac{z + l_1v + l_2}{cv + d}; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad (l_1, l_2) \in \mathbb{R}^2. \quad (45)$$

Proof. Let us use the notation of [24]. We denote $G^J(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \rtimes H(\mathbb{R})$, where here $H(\mathbb{R})$ denotes the real HW group with the composition law:

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = \left(\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \begin{vmatrix} X \\ X' \end{vmatrix} \right), \quad \begin{vmatrix} X \\ X' \end{vmatrix} = \det \begin{pmatrix} X \\ X' \end{pmatrix}. \quad (46)$$

If $g = (M, X, \kappa) \in G^J(\mathbb{R})$, where $M \in \text{SL}_2(\mathbb{R})$, $X = (\lambda, \mu)$, $(X, \kappa) \in \mathbb{R}^3$, then the composition law in the real Jacobi group is

$$gg' = \left(MM', XM' + X', \kappa + \kappa' + \begin{pmatrix} XM' \\ X' \end{pmatrix} \right). \quad (47)$$

The action of $G^J(\mathbb{R})$ over the $H(\mathbb{R})$ is

$$M(X, \kappa)M^{-1} = (XM^{-1}, \kappa). \quad (48)$$

Let us consider the Iwasawa decomposition for a matrix $M \in \text{SL}_2(\mathbb{R})$:

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad y > 0. \quad (49)$$

If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (50)$$

then we find for x, y, θ in (49)

$$x = \frac{ac + bd}{d^2 + c^2}; \quad y = \frac{1}{d^2 + c^2}; \quad \sin \theta = -\frac{c}{\sqrt{c^2 + d^2}}; \quad \cos \theta = \frac{d}{\sqrt{c^2 + d^2}}. \quad (51)$$

For $g = (M, X, \kappa) \in G^J(\mathbb{R})$, the EZ-coordinates (Eichler-Zagier, cf. the definition at p. 12 and p. 51 in [24]) are $(x, y, \theta, \lambda, \mu, \kappa)$. Let $\tau = x + iy \in \mathcal{H}_1$, $z = \xi + i\eta = p\tau + q$, where

$$(p, q) = XM^{-1} = (\lambda d - \mu c, -\lambda b + \mu a). \quad (52)$$

If we attache a “*” to the results of elements of the composition rule (47), we have

$$x_* = \frac{AC + BD}{D^2 + C^2}; \quad y_* = \frac{1}{D^2 + C^2}, \quad (53)$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \quad (54)$$

We find out:

$$D^2 + C^2 = c^2(a'^2 + b'^2) + d^2(c'^2 + d'^2) + 2cd(a'c' + b'd'),$$

i.e.

$$D^2 + C^2 = c^2(a'^2 + b'^2) + \frac{d^2}{y'} + 2cd \frac{x'}{y'}.$$

Similarly,

$$AC + BD = ac(a'^2 + b'^2) + (ad + bc)\frac{x'}{y'} + \frac{bd}{y'}.$$

We find for $\tau_* = x_* + iy_*$

$$\tau_* = \frac{ac(a'^2 + b'^2)y' + (ad + bc)x' + iy' + bd}{c^2(a'^2 + b'^2)y' + 2cdx' + d^2}. \quad (55)$$

Let us verify the first relation (45), in the present notation

$$\tau_* = \frac{a\tau' + b}{c\tau' + d}, \quad (56)$$

where

$$\tau' = x' + iy' = \frac{a'c' + b'd' + i}{d'^2 + c'^2}. \quad (57)$$

Combining (56), (57), we find out

$$\tau_* = \frac{(ax' + b)(cx' + d) + acy'^2 + iy'}{(cx' + d)^2 + c^2y'^2}, \quad (58)$$

and we have to verify the identify (55) and (58).

In order to prove the second equation (45), we calculate firstly

$$(P_*, Q_*) = (LD - MC, -LB + MA),$$

where

$$(L, M) = (\lambda' + \lambda a' + \mu c', \mu' + \lambda b' + \mu d'),$$

and we find

$$P_* = \lambda'(cb' + dd') - \mu'(ca' + dc') + \lambda d - \mu c; \quad (59a)$$

$$Q_* = -\lambda'(ab' + bd') + \mu'(ad' + bc') - \lambda b + \mu a. \quad (59b)$$

Then we obtain

$$z_* := P_*\tau_* + Q_* = \frac{(P_*a + Q_*c)\tau' + P_*b + Q_*a}{c\tau' + d}.$$

The nominator E of the last expression of τ^* should be identified with $E = p'\tau' + q' + \lambda\tau' + \mu$, *i.e.* it remains to verify that

$$P_*a + Q_*b = p' + \lambda;$$

$$P_*b + Q_*d = q' + \mu'.$$

In conclusion, using the multiplication law (47), the Iwasawa decomposition (49) and the equations (51), (52), we have obtained the action of $G^J(\mathbb{R})$ on the base \mathfrak{X}_1^J

$$g(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right), \quad (60)$$

and Remark 1 is proved.

Let us now recall that

$$C^{-1}\mathrm{SL}_2(\mathbb{R})C = \mathrm{SU}(1, 1), \quad \text{where } C = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}. \quad (61)$$

If $M \in \mathrm{SL}_2(\mathbb{R})$ is the matrix (50), then, under the transformation (61)

$$M_* = C^{-1}MC = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1, \quad (62)$$

where

$$2\alpha = a + d + i(b - c); \quad 2\beta = a - d - i(b + c). \quad (63)$$

Now we pass to the complex group $G_{\mathbb{C}}^J = C^{-1}G^J(\mathbb{R})C$. Under the transformation (61), $g = (M, X, \kappa) \in \mathrm{SL}_2(\mathbb{R}) \times H(\mathbb{R})$ is twisted to $g_* = (M_*, X_*, \kappa)$, where M_* is given by (62), while, due to action (48), $X_* = XC = (i\lambda - \mu, i\lambda + \mu)$.

Also the map (61) induces a transformation of the bounded domain \mathcal{D}_1 into the upper half plane \mathcal{H}_1 and

$$\tau \in \mathcal{H}_1 \mapsto \tau_* = C^{-1}(\tau) = \frac{\tau - i}{\tau + i} \in \mathcal{D}_1. \quad (64)$$

The action $C^{-1}G_0^J(\mathbb{R})C$ descends on the basis to the biholomorphic map: $\check{C}^{-1} : \check{\mathcal{X}}_1^J := \mathcal{H}_1 \times \mathbb{C} \rightarrow \mathcal{D}_1^J := \mathcal{D}_1 \times \mathbb{C} : (\tau, z) \mapsto (\tau_*, z_*)$. Here τ_* is given by (64), while $z_* = p_*\tau_* + q_*$. So, $(p, q) = (\lambda, \mu)M^{-1}$, and $(p_*, q_*) = (\lambda_*, \mu_*)M_*^{-1}$. But $M_* = C^{-1}MC$, and $(p_*, q_*) = (p, q)C = (-q + ip, q + ip)$, and we get $z_* = \frac{2iz}{\tau + i}$.

In a different notation, we have shown that

Remark 2. The action $C^{-1}G_0^J(\mathbb{R})C$, descends on the basis to the biholomorphic map: $\check{C}^{-1} : \check{\mathcal{X}}_1^J := \mathcal{H}_1 \times \mathbb{C} \rightarrow \mathcal{D}_1^J := \mathcal{D}_1 \times \mathbb{C} :$

$$w = \frac{v - i}{v + i}; \quad z = \frac{2iu}{v + i}, \quad w \in \mathcal{D}_1, \quad v \in \mathcal{H}_1, \quad z \in \mathbb{C}. \quad (65)$$

The $G_0^J(\mathbb{R})$ -invariant closed two-form considered by Kähler-Berndt is:

$$\omega'_1 = \alpha \frac{dv \wedge d\bar{v}}{(v - \bar{v})^2} + \beta \frac{1}{v - \bar{v}} B \wedge \bar{B}, \quad B = du - \frac{u - \bar{u}}{v - \bar{v}} dv, \quad v, u \in \mathbb{C}, \quad \Im(v) > 0, \quad (66)$$

cf. §36 in [27]; see also §3.2 in [23].

Under the mapping (65), the two-form ω_1 (44) reads

$$-i \omega'_1 = -\frac{2k}{(\bar{v} - v)^2} dv \wedge d\bar{v} + \frac{2}{i(\bar{v} - v)} B \wedge \bar{B}, \quad (67)$$

i.e. (66). In fact, we have proved that

Remark 3. *When expressed in the coordinates $(v, u) \in \mathcal{X}_1^J$ which are related to the coordinates $(w, z) \in \mathcal{D}_1^J$ by the map (65) given by Remark 2, the Kähler two-form (44) is identical with the one (67) considered by Kähler-Berndt.*

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REFERENCES

1. A. M. Perelomov, *Generalized Coherent States and their Applications*, Springer, Berlin, 1986.
2. W.-M. Zhang, D. H. Feng, R. Gilmore, *Rev. Mod. Phys.* **62**, 867–927 (1990).
3. S. T. Ali, J.-P. Antoine, J.-P. Gazeau, *Coherent states, wavelets, and their generalizations*, Springer-Verlag, New York, 2000.
4. S. Sivakumar, *J. Opt. B: Quantum Semiclass. Opt.* **2**, R61–R75 (2000).
5. V. V. Dodonov, *J. Opt. B.: Quantum semiclassical Opt.* **4**, R1–R33 (2002).
6. S. Berceanu, *Rev. Math. Phys.* **18**, 163–199 (2006).
7. S. Berceanu, *J. Geom. Symmetry Phys.* **5**, 5–13 (2006);
S. Berceanu, *A holomorphic representation of Jacobi algebra in several dimensions*, arXiv: math.DG/060404381;
8. S. Berceanu, *Coherent states associated to the Jacobi group – a variation on a theme by Erich Kähler*, communication at the XXV Workshop of Geometric Methods in Physics, Białowieża, Poland, June 2 – July 8, 2006.
9. K. B. Wolf, *The Heisenberg-Weyl ring in quantum mechanics*, in *Group theory and its applications*, Vol. 3, Ed. E. M. Loebl, Academic Press, New York 189–247, (1975).
10. V. Guillemin, S. Sternberg, *Geometric Asymptotics*, American Mathematical Society, Providence, R. I. 1977.
11. V. Guillemin, S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.
12. H. Bacry, M. Cadilhac, *Phys. Rev.* **A23**, 2533–536 (1981).
13. D. Stoler, *J. Opt. Soc. Am.* **A71**, 334–341 (1981).
14. M. Nazarathy, J. Shamir, *J. Opt. Soc. Am.* **70**, 150–158 (1980).
15. M. Nazarathy, J. Shamir, *J. Opt. Soc. Am.* **72**, 356–364 (1982).
16. R. Simon, K. B. Wolf, *J. Opt. Soc. Am.* **A17**, 342–355 (2000).
17. P. D. Drummond, Z. Ficek, Eds., *Quantum squeezing*, Springer, Berlin, 2004.
18. M. Eichler, D. Zagier, *The theory of Jacobi forms*, *Progress in Mathematics*, **55**, Birkhäuser, Boston, MA, 1985.
19. U. Niederer, *Helv. Phys. Acta.* **45**, 802–810 (1972/73).
20. K. B. Wolf, *Integral transforms in science and engineering*, Plenum Publ. Corp. New York, 1979.
21. K. B. Wolf, *Geometric Optics on Phase Space*, Springer, Berlin (2004).
22. C. L. Siegel, *Symplectic geometry*, Academic Press, New York, 1964.

22. I. Satake, *Algebraic structures of symmetric domains*, Publ. Math. Soc. Japan **14**, Princeton Univ. Press, 1980.
23. R. Berndt, *Sur l'arithmétique du corps des fonctions elliptiques de niveau N* , in *Seminar on number theory*, Paris 1982–83, Progr. Math., **51**, Birkhäuser Boston, Boston, MA (1984) 21–32.
24. R. Berndt, R. Schmidt, *Elements of the representation theory of the Jacobi group*, Progress in Mathematics, 163, Birkhäuser Verlag, Basel, 1998.
25. E. Kähler, Rend. Sem. Mat. Fis. Milano **53**, 359–90 (1983)
26. E. Kähler, *The Poincaré group*, in *Clifford algebras and their applications in mathematical physics* (Canterbury, 1985), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **183**, Reidel, Dordrecht, 265–272 (1986).
27. E. Kähler, Rend. Accad. Naz. Sci. XL Mem. Mat. **5**, 115–177 (1992).
28. R. Simon, E. C. G. Sudarshan, N. Mukunda, Phys. Rev. **A37**, 3028–3038 (1988).
29. S. Berceanu, A. Gheorghe, J. Math. Phys. **34**, 998–1007 (1992).
30. S. Berceanu, L. Boutet de Monvel, J. Math. Phys. **34**, 2353–2371 (1993).
31. S. Berceanu, *Realization of coherent state algebras by differential operators*, in *Advances in Operator Algebras and Mathematical Physics*, Edited by F. Boca, O. Bratteli, R. Longo, H. Siedentop, The Theta Foundation, Bucharest, 1–24 (2005).
32. W. Lisiecki, Rep. Math. Phys. **35**, 327–358 (1995).
33. K.-H. Neeb, *Holomorphy and Convexity in Lie Theory*, de Gruyter Expositions in Mathematics 28, Walter de Gruyter, Berlin-New York, 2000.
34. P. Stoler, Phys. Rev. **D1**, 3217–3219 (1970); —, II, Phys. Rev. **D4**, 1925–1926 (1971).
35. V. Bargmann, Ann. of Math. **48**, 568–640 (1947).
36. R. Berndt, S. Böcherer, Math. Z. **204**, 13–44 (1990).
37. R. Berndt, *Coadjoint orbits and representations of the Jacobi group*, IHES/M/03/37 (2003).
38. *Erich Kähler: Mathematische Werke; Mathematical Works*, Edited by R. Berndt, O. Riemenschneider, Walter de Gruyter, Berlin–New York, 2003.
39. R. Simon, N. Mukunda, Opt. Commun. **95**, 39–45 (1993).
40. M. Kogelnik, Bell Sys. Tech. J. **44**, 455–494 (1965);
H. Kogelnik, T. Li, Proc. IEEE **54**, 1312–1328 (1966);
J. Arnaud, H. Kogelnik, Appl. Optics **5**, 1687–1695 (1969);
J. J. Arnaud, *Beam and fiber optics*, Academic Press, 1976.