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MATHEMATICAL AND GENERAL PHYSICS

MOON'S PROBLEM

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Abstract. A series expansion in powers of eccentricities is set up for the motion in central-field potentials, which connects such movements to anharmonic oscillators. The method is used to rederive the solution of Kepler's problem, describe the motion of a particle in a general central-field potential, and discuss the closed orbits and the first sign of chaos. Similarly, Moon's problem is tackled by the same method, where the motion turns out to be described by triple series in powers of eccentricities, inclination against the ecliptic and the gravitational interaction originating in the Sun. Newton's results are thus rederived in the first-order of the perturbation theory, regarding the "four Moons" and four periodicities, as known as early as from the classical Greeks. The method can also be applied to the Jupiter-Saturn couple, where their mutual gravitational interaction may be viewed as a perturbation. The missing integrals of motion, Poincare's "weak chaos" and trajectory strong-chaotical instabilities are also introduced. Finally, a new route of quantizing the motion in central-field potentials is presented, as based on the eccentricities expansion.

Key words: Moon's problem.

INTRODUCTION

The Moon's motion in the sky was recorded from ancient times. The four periodicities associated with this motion were known ca 3,000 years ago, with one second of time accuracy (which means five decimals, and 1 km in position), and are the first historical measurements in Natural Sciences. Newton's Natural Philosophy was obviously motivated by the Planetary System of Celestial Bodies, like the Earth rotating about the Sun, and, firstly, the Moon's motion as the only one amenable to accurate empirical observations. Mathematical Physics was born from Celestial Mechanics, and the great mathematical tools developed by mathematicians like Euler, Lagrange, Hamilton, Jacobi, etc. of the 18th and 19th centuries were occasioned by the three-body problem like the Sun, the Earth and the Moon. It was this problem where Poincaré noticed particular signs of complex motion which later became known as "chaos". The three-body problem, ergodic hypothesis and anharmonic oscillators are the three basic pillars that illustrate a chaotical behaviour. In particular, endless trajectories that never repeat, sensitivity upon the initial conditions, non-linearity, instabilities, bifurcation, fractal dimensions, etc. are all typical of “chaotical issues”. The Apollo program of aselenization was based on such calculations of the three-body complex Sun-Earth-Moon, and modern computers brought new insights into such complex movements.

KEPLER’S PROBLEM

Kepler’s problem is the description of the motion of a particle of mass m in the gravitational potential $-\alpha/r$, where $\alpha > 0$. Like any other central-field potential, the gravitational potential conserves the angular momentum **L**, so the motion is confined to a plane, and

$$L = mr^2\dot{\varphi}, \tag{1}$$

where φ is the angular coordinate. Equation (1) shows that the motion sweeps equal areas in equal times (Kepler’s second law).

The energy of the motion reads

$$E = m\dot{r}^2/2 + mr^2\dot{\varphi}^2/2 - \alpha/r = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r, \tag{2}$$

as if the particle moves in an effective potential

$$U = L^2/2mr^2 - \alpha/r, \tag{3}$$

exhibiting the centrifugal $1/r^2$ - energy. The orbits proceed between $r_1 = a(1-e)$ and $r_2 = a(1+e)$, where $a = \alpha/2|E|$ and

$$e = \sqrt{1 - 2L^2|E|/m\alpha^2} \tag{4}$$

is the eccentricity, for negative energies above $E_{min} = -m\alpha^2/2L^2$.

The effective potential (3) reaches its minimum value E_{min} for

$$r_0 = L^2/m\alpha, \tag{5}$$

where the eccentricity vanishes and the orbit is circular with radius r_0. By (4), the energy can also be represented as

$$|E| = \frac{\alpha}{2r_0}(1-e^2), \tag{6}$$

or $r_0 = a(1-e^2)$.

The expansion of the effective potential U given by (3) around its minimum value gives

$$U = -\frac{\alpha}{2r_0} + \frac{\alpha}{2r_0^3}(r - r_0)^2 - \frac{\alpha}{r_0^4}(r - r_0)^3 + \ldots, \tag{7}$$

i.e., a small-oscillations expansion valid for $|r - r_0| \ll r_0$.

It is convenient to write $r - r_0 = Au$, where u is dimensionless, and cast the energy given by (2), (3) and (7) into the form

$$E = -\alpha/2r_0 + mA^2\dot{u}^2/2 + (\alpha A^2/2r_0^3)u^2 - (\alpha A^3/r_0^4)u^3 + \ldots, \tag{8}$$

or

$$E = -\alpha/2r_0 + mA^2[\dot{u}^2/2 + \omega^2 u^2/2 - (A/r_0)\omega^2 u^3 + \ldots], \tag{9}$$

where $\omega^2 = \alpha/mr_0^3$ and $A/r_0 = \varepsilon$ can be viewed as a small perturbation parameter. Equation (9) can also be written as

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\omega^2 u^3 + \ldots), \tag{10}$$

which tells that the eccentricity e is related to the perturbation parameter ε. Equation (9) leads to the motion

$$\ddot{u} + \omega^2 u - 3\varepsilon\omega^2 u^2 + \ldots = 0 \tag{11}$$

of an anharmonic oscillator. Within the harmonic approximation the solution of equation (11) can be represented as $u^{(0)} = -\cos\omega t$, and

$$r^{(0)} = r_0 - A\cos\omega t. \tag{12}$$

The amplitude A can be derived from energy $E = -\alpha/2r_0 + mA^2\omega^2/2$ given by (9) or, equivalently, from equation (10). It leads to

$$\varepsilon = A/r_0 = e \ll 1, \tag{13}$$

i.e., the eccentricity e of the orbit is the ratio ε of the amplitude A of the harmonic oscillation to the original orbit radius r_0. The small-oscillations treatment is valid for small eccentricities.

Therefore, the solution of the motion given by (12) can be written as

$$r^{(0)} = r_0(1 - e\cos\omega t), \tag{14}$$

and, by (1),[1]

$$\varphi = \omega t + 2e\sin\omega t. \tag{15}$$

[1] Noteworthy, $L = \omega I$, where $I = mr_0^2$ is the moment of inertia.

It describes a circular motion, shifted by r_0e. Indeed, $x = r_0e + r^{(0)}\cos\varphi$ and $y = r^{(0)}\sin\varphi$, such that $x^2 + y^2 = r_0^2$ within the harmonic approximation. In addition, $\omega^2 = \alpha/mr_0^3$ shows that the square of the motion period is proportional to the third power of the linear size of the orbit (Kepler's third law).[2] By (15), $\omega t = \varphi - 2e\sin\varphi$.

The first-order cubic correction to equation (11) leads to

$$u = u^{(0)} + \varepsilon u^{(1)} = -\cos\omega t - \varepsilon\cos\omega t + \frac{\varepsilon}{2}(3 - \cos 2\omega t), \tag{16}$$

and equation (10) gives $\varepsilon = e(1-e)$. The corresponding radius reads

$$r = r_0\left[1 - e\cos\omega t + \frac{e^2}{2}(3 - \cos 2\omega t)\right], \tag{17}$$

which, by (1), leads to

$$\varphi = \omega t + 2e\sin\omega t - \frac{e^2}{2}(3\omega t - \frac{5}{2}\sin 2\omega t)]. \tag{18}$$

Equation (18) can easily be inverted to give

$$\omega t = \varphi - 2e\sin\varphi + \frac{3e^2}{2}(\varphi + \frac{1}{2}\sin 2\varphi), \tag{19}$$

which transforms (17) into

$$r = r_0(1 - e\cos\varphi + e^2\cos^2\varphi + ...)\ . \tag{20}$$

Within this approximation, equation (20) describes an ellipse,

$$r/r_0 = 1 - e\cos\varphi + e^2\cos^2\varphi + ... = 1/(1 + e\cos\varphi), \tag{21}$$

with the semi-major axis $a = r_0/(1-e^2) = r_0(1 + e^2 + ...)$, the semi-minor axis $b = r_0/(1-e^2)^{1/2} = r_0(1 + e^2/2 + ...)$ and the origin displaced by $ae = r_0e + ...$ in the focus ae (Kepler's first law).[3]

According to equation (19) the period T of the motion is given by

$$\omega T = 2\pi(1 + 3e^2/2), \tag{22}$$

which shows that frequency ω is shifted to $\Omega = \omega(1 - 3e^2/2) = (\alpha/ma^3)^{1/2}$. The frequency shift $\Delta\omega/\omega = -3e^2/2$ ensures the cancellation of the resonant contributions to the second-order cubic correction and first-order quartic correction to the anharmonic motion.

[2] J. Kepler, *Harmonices Mundi*, Linz (1619).

[3] Indeed, from (21), $\cos\varphi = x/(r_0 - ex)$ and $\sin\varphi = y/(r_0 - ex)$, hence the ellipse equation.

GENERAL CENTRAL-FIELD POTENTIAL

Let $v(r)$ be an attractive central-field potential, such that the radial motion proceeds between r_1 and r_2 given by[4]

$$L^2/2mr_{1,2}^2 + v(r_{1,2}) = E < 0. \tag{23}$$

The effective potential $U(r) = L^2/2mr^2 + v(r)$ has a minimum value $-u_0 = = r_0 v_1(1/2 + v_0/r_0 v_1) < 0$ for r_0 given by $L^2 = mr_0^3 v_1$, where v_0, v_1, v_2, ... denote the potential and, respectively, its derivatives for r_0. Making use of $r - r_0 = Au$ and $A/r_0 = \varepsilon$, the energy E can be written as

$$E = -u_0 + mA^2[\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\beta\omega^2 u^3 + \varepsilon^2\gamma\omega^2 u^4 ...], \tag{24}$$

where $m\omega^2 = 3v_1/r_0 + v_2$, $\beta = (2v_1 - r_0^2 v_3/6)/(3v_1 + r_0 v_2)$ and $\gamma = (5v_1/2 + +r_0^3 v_4/24)/(3v_1 + r_0 v_2)$. Making use of the eccentricity e defined by $e^2 = = \delta(1 - |E|/u_0)$, where $\delta = -(v_1 + 2v_0/r_0)/(3v_1 + r_0 v_2)$, equation (24) can be rewritten as

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\beta\omega^2 u^3 + \varepsilon^2\gamma\omega^2 u^4 + ...). \tag{25}$$

The equation of motion given by (24) reads

$$\ddot{u} + \omega^2 u - 3\varepsilon\beta\omega^2 u^2 + 4\varepsilon^2\gamma\omega^2 u^3 ... = 0, \tag{26}$$

and its solution is given by

$$r = r_0\left[1 - e\cos\omega t + \frac{\beta e^2}{2}(3 - \cos 2\omega t)\right], \tag{27}$$

to the first-order of the cubic anharmonicity, where $e = \varepsilon(1 + \beta\varepsilon)$. Similarly, the angular variable is given by

$$\varphi = \sqrt{v_1/(3v_1 + r_0 v_2)}\left\{\omega t + 2e\sin\omega t - \frac{e^2}{2}\left[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2}\sin 2\omega t\right]\right\}. \tag{28}$$

One can see that, in general, the trajectory of the motion is not closed, except for

$$\sqrt{v_1/(3v_1 + r_0 v_2)} = p/q \tag{29}$$

[4] In order to avoid the fall on the centre the potential $v(r)$ must be less singular at the origin than $-L^2/2mr^2$.

where p/q is a simple fraction. The gravitational potential $v(r)=-\alpha/r$ gives $p/q=1$, while the spatial-oscillator potential $v(r)=const+\alpha r^2$ gives $p/q=1/2$ ($\beta=1/2$, $\gamma=5/8$).

Denoting $1/\nu=\sqrt{v_1/(3v_1+r_0v_2)}$ and introducing the new phase $\chi=\nu\varphi$, equation (28) can be rewritten as

$$\chi=\omega t+2e\sin\omega t-\frac{e^2}{2}\left[3(2\beta-1)\omega t-\frac{2\beta+3}{2}\sin 2\omega t\right], \quad (30)$$

and it can be easily inverted to give

$$\omega t=\chi-2e\sin\chi+\frac{e^2}{2}\left[3(2\beta-1)\chi-\frac{2\beta-5}{2}\sin 2\chi\right]. \quad (31)$$

Making use of (31) the equation of the trajectory (27) becomes

$$r=r_0'[1-e\cos\chi+(2-\beta)e^2\cos^2\chi], \quad (32)$$

where $r_0'=r_0[1-2(1-\beta)e^2]$. For the gravitational potential $\beta=1$, and equation (21) is recovered from (32), while for the spatial oscillator $\beta=1/2$, $\chi=2\varphi$ and (32) becomes

$$r=r_0'\left[1-e\cos 2\varphi+\frac{3e^2}{2}\cos^2 2\varphi\right]. \quad (33)$$

Since (33) is equivalent to $r^2={r_0}^2/(1+2e\cos 2\varphi)$, it is easy to see that the corresponding trajectory is an ellipse centered at the origin. One can see from (30) that the spatial oscillator does not shift the frequency, but reduces it instead to $\omega/2$.

CLOSED ORBITS AND THE FIRST SIGN OF "CHAOS"

Higher-order contributions of the anharmonicities may lead, in general, to a shift in frequency, in order to avoid, at each step of the perturbation calculations, the resonant terms.[5] Equation (27) for the radius gets thereby a shifted frequency ω', and equation (1) for the phase motion reads now $\dot{\varphi}=(L/m\omega r^2)(\omega/\omega')\omega'$, which changes, in general, the prefactor $1/\nu$ in equation (28).[6] This is valid as long as the

[5] Such terms are also called "secular terms", and the shift in frequency is also known as the Poincaré-Lindstedt expansion (H. Poincaré, *Les Methodes Nouvelles de la Mecanique Celeste*, Gauthier-Villars, Paris (1892); A. Lindstedt, *Uber die Integration einer fur die Storungstheorie wichtigen Differentialgleichung*, Astron. Nach. **103**, 211 (1882)).

[6] Equation (29) is the first term of the series expansion of the well-known closure condition

$$\Delta\varphi/2\pi=(1/\pi)\int_{r_1}^{r_2}dr\cdot(L/r^2)/\sqrt{3m(E-v(r))-L^2/r^2}=p/q.$$

calculations are confined to finite orders of perturbation series, as for small oscillations and eccentricities, for instance. In the limit of the series summation the orbits are closed only for two power-law potentials: the gravitational potential $-\alpha/r$ and the spatial-oscillator potential $const+\alpha r^2$. Indeed, this can be seen easily on the equation of motion for the trajectory $r(\varphi)$, as given by (1) and (2), whose integration requires a quadratic form of the integrand, the only one able to lead to circular functions.[7] In general, the trajectories are closed provided the potentials are such as to cancel recurssively the frequency shifts in the formal perturbation series. However, for sufficiently large p and q, and a large number of cycles, the orbits are practically closed for any potential.[8] This is another illustration of the "ergodic hypoyhesis", and is viewed sometimes as the first sign of "chaos" and "chaotical" behaviour.

MOON'S PROBLEM

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of two bodies of mass m_1 (Earth, $m_1 \simeq 6\times 10^{24}\,\mathrm{Kg}$) and, respectively, m_2 (Moon, $m_2 \simeq 7\times 10^{22}\,\mathrm{Kg}$), subjected to gravitational potentials $-Gm_0m_1/r_1$, $-Gm_0m_2/r_2$ and interacting through $-Gm_1m_2/\left|\mathbf{r}_1-\mathbf{r}_2\right|$, where $G\simeq 6.7\times 10^{-11}\,\mathrm{m}^3/\mathrm{Kg}\cdot\mathrm{s}^2$ is the gravitational constant. The body of mass m_0 (Sun, $m_0 \simeq 2\times 10^{30}\,\mathrm{Kg}$) is at rest. The energy is given by

$$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/\left|\mathbf{r}_1-\mathbf{r}_2\right|, \qquad (34)$$

and the angular momentum reads

$$\mathbf{L}_{tot} = m_1\mathbf{r}_1\times\dot{\mathbf{r}}_1 + m_2\mathbf{r}_2\times\dot{\mathbf{r}}_2\,. \qquad (35)$$

It is easy to see that $\mathbf{L}_{tot}$ is conserved. However, there are no other constants of motion (at least analytical),[9] and, consequently, the problem is termed "non-integrable". Making use of the center-of-mass coordinate $\mathbf{R}=m_1\mathbf{r}_1/M+$ $+m_2\mathbf{r}_2/M$, where $M=m_1+m_2$, and the relative coordinate $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, the angular momentum becomes

[7] Making use of the substitution $r=1/u$ the equation for the trajectory $u(\varphi)$ reads $u''+u=-(m/L^2)\partial v/\partial u$, whose solution is given by circular functions only for the gravitational potential $v\sim u$ and the spatial oscillator potential $v\sim 1/u^2$. This observation is called sometime "Bertrand's theorem" (J. Bertrand, *Mecanique Analytique*, Comptes Rendus, Acad. Sci. **77** 849 (1873)).

[8] To the extent to which an irrational number is approximated by a rational number.

[9] This is sometime referred to as a Bruns-Poincaré theorem.

$$\mathbf{L}_{tot} = M\mathbf{R}\times\dot{\mathbf{R}} + m\mathbf{r}\times\dot{\mathbf{r}}, \tag{36}$$

where $m = m_1 m_2/M$ is the relative mass. Similarly, the energy can be written as

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - Gm_0 m_1/\left|\mathbf{R} - m_2\mathbf{r}/M\right| - Gm_0 m_2/\left|\mathbf{R} + m_1\mathbf{r}/M\right| - Gm_1 m_2/r. \tag{37}$$

Since $r \ll R$ (Sun-Earth distance $r_1 \simeq 15\times 10^7$ Km, Moon-Earth distance $r \simeq 380\,000$ Km) it is convenient to expand the gravitational potentials in (37) in powers of $\mathbf{rR}/R^2$. Keeping only the quadrupolar contribution the energy becomes

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - \alpha/R - \beta/r - \gamma[3(\mathbf{rR})^2/R^2 - r^2]/R^3, \tag{38}$$

where $\alpha = Gm_0 M$, $\beta = GmM$ and $\gamma = Gm_0 m/2$, or

$$E = E_1 + E_2 + \gamma v, \tag{39}$$

where

$$E_1 = M\dot{\mathbf{R}}^2/2 - \alpha/R, \qquad E_2 = m\dot{\mathbf{r}}^2/2 - \beta/r, \tag{40}$$

and

$$v = -r^2(3\cos^2\chi - 1)/R^3. \tag{41}$$

The angle χ in (41) is the angle between the two vectors $\mathbf{r}$ and $\mathbf{R}$. Since $r/R \sim 3\times 10^{-3}$ for Moon-Earth-Sun (and $(r/R)^2 \sim 10^{-5}$) the interaction v may be viewed as a small perturbation, and γ in (39) may act as a formal perturbation parameter. The solutions of the equations of motion corresponding to (39) are written in the generic form $u = u^{(0)} + \gamma u^{(1)} + \ldots$. It is convenient now to employ polar coordinates and rewrite (40) as

$$E_1 = M\dot{R}^2/2 + MR^2(\dot{\Theta}^2 + \dot{\Phi}^2\sin^2\Theta)/2 - \alpha/R, \tag{42}$$

and, similarly,

$$E_2 = m\dot{r}^2/2 + mr^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)/2 - \beta/r, \tag{43}$$

where $\cos\chi = \sin\Theta\sin\theta\cos(\Phi - \varphi) + \cos\Theta\cos\theta$ in (41). The angular momentum of the relative motion reads

$$l_x = -mr^2(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi), \quad l_y = mr^2(\dot{\theta}\cos\varphi - \dot{\varphi}\sin\theta\cos\theta\sin\varphi),$$
$$l_z = mr^2\dot{\varphi}\sin^2\theta, \tag{44}$$

or $l_r = 0$, $l_\theta = -mr^2\dot{\varphi}\sin\theta$, $l_\varphi = mr^2\dot{\theta}$. Similar expressions hold for the angular momentum $\mathbf{L}$ of the center of mass, and $\mathbf{L}_{tot} = \mathbf{L} + \mathbf{l}$.[10]

[10] In view of the great disparity between m_1 and m_2, the center of mass is located practically on the first body (Earth, $M \simeq m_1$), and the relative motion corresponds practically to the second body (Moon, $m \simeq m_2$).

PERTURBATION THEORY

The ratio of the perturbation γv to energy α/R is of the order of 10^{-7}. Consequently, the motion of the M-body can be considered as being unperturbed, and described by an independent Kepler's problem of the form given by (17) and (18), where the frequency is given by $\Omega^2 = \alpha/MR_0^3$,[11] the parameter R_0 is given by $R_0 = L^{(0)2}/M\alpha$ (where $L^{(0)}$ is the unperturbed angular moment of the M-body motion) and the eccentricity e_1 is given by $|E_1| = \frac{\alpha}{2R_0}(1-e_1^2)$. Similarly, the ratio of the perturbation γv to energy β/r is of the order of 10^{-3}, and first-order corrections in γ are retained in the m-body motion. Its zeroth order trajectory is described by equations similar with (17) and (18), where the frequency ω is given by $\omega^2 = \beta/mr_0^3$,[12] the parameter r_0 is given by $r_0 = l^{(0)2}/m\beta$ ($l^{(0)}$ being the unperturbed angular momentum of the m-body motion) and eccentricity e_2 is given by $|E_2| = \frac{\beta}{2r_0}(1-e_2^2)$. In addition, in order to preserve the generality, the unperturbed m-body orbit must be rotated both by an angle φ_0 (about the z-axis) and by an angle θ_0 (about one of the x- or y-axis). The latter gives the inclination of the m-orbit with respect to the plane of the M- body orbit.[13] The former (φ_0-) rotation can be accounted for by changing the initial moment of time. The θ_0-rotation (about the x-axis) leads to the new coordinates $r' = r$, and θ', φ' given by

$$\cos\theta' = \sin\theta_0 \sin\varphi, \quad \tan\varphi' = \cos\theta_0 \tan\varphi. \tag{45}$$

One can check easily that $(d\theta'/d\varphi)^2 + (d\varphi'/d\varphi)^2 \sin^2\theta' = 1$, which expresses the conservation of the angular momentum under this rotation. Equations (45) lead to

$$\begin{aligned} \varphi' &= \varphi - \frac{1}{4}\theta_0^2 \sin 2\varphi + \ldots = \omega t + 2e_2 \sin\omega t - \frac{1}{4}\theta_0^2 \sin 2\omega t + \ldots, \\ \theta' &= \pi/2 - \theta_0 \sin\varphi + \ldots = \pi/2 - \theta_0 \sin\omega t + \ldots, \\ r' &= r = r_0(1 - e_2 \cos\omega t + \ldots). \end{aligned} \tag{46}$$

These are the zeroth order contributions $u^{(0)}$ to the general solution $u = u^{(0)} + \gamma u^{(1)} + \ldots$. for the m-body motion. One can check easily that they do indeed verify the unperturbed equations of motion.

[11] From $\Omega^2 = \alpha/MR_0^3$ one can check easily the Earth's year ~ 365 days.

[12] Moon's period ~ 27 days is checked from $\omega^2 = \beta/mr_0^3$.

[13] It corresponds to Moon's orbit inclination against the ecliptic, which is approximately $\theta_0 = 5° = \pi/36$.

EQUATIONS OF MOTION

The equations of motion for the m-body, as given by (39) to (41), read

$$\begin{aligned} &m\ddot{r} - mr(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \beta/r^2 = 2\gamma(r/R^3)(3\cos^2\chi - 1),\\ &d(mr^2\dot{\theta})/dt - mr^2\dot{\varphi}^2 \sin\theta\cos\theta =\\ &= 6\gamma(r^2/R^3)\cos\chi[\sin\Theta\cos\theta\cos(\Phi-\varphi) - \cos\Theta\sin\theta],\\ &d(mr^2\sin^2\theta\dot{\varphi})/dt = 6\gamma(r^2/R^3)\cos\chi\sin\Theta\sin\theta\sin(\Phi-\varphi). \end{aligned} \tag{47}$$

To the lowest order of perturbation theory the coordinates $R = R_0$, $\Theta = \pi/2$, $\Phi = \Omega t$ and $r = r_0$, $\theta = \pi/2$, $\varphi = \omega t$ are inserted in the *rhs* of (47), and only the linear terms in eccentricity e_2 and quadratic in inclination angle θ_0 are retained.[14] Within this approximation the m-body motion reduces to a Kepler's problem in an external field.[15] In addition, Ω may be droped out in comparison with ω, since $\Omega \ll \omega$.[16] Doing so, equations (47) become

$$\begin{aligned} &m\ddot{r} - mr(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \beta/r^2 = \gamma(r_0/R_0^3)(1 + 3\cos 2\omega t),\\ &d(mr^2\dot{\theta})/dt - mr^2\dot{\varphi}^2 \sin\theta\cos\theta = 0,\\ &d(mr^2\sin^2\theta\dot{\varphi})/dt = -3\gamma(r_0^2/R_0^3)\sin 2\omega t. \end{aligned} \tag{48}$$

The solutions of these equations are looked for in the form $r = r' + \gamma r_1 + ...$, $\theta = \theta' + \gamma\theta_1 + ...$ and $\varphi = \varphi' + \gamma\varphi_1 + ...$, where r', θ' and φ' are given by (46). The presence of the constant term in the first equation (48) gives rise to secular terms, so the frequency ω is renormalized to ω' in the zeroth order solutions given by (46). This renormalization implies a shift in frequency of the order of γ, which, as it is well known,[17] is computed by requiring the cancellation of the secular terms.

It is easy to see that equation on the third row in (48) leads to the integral of motion

$$mr^2\dot{\varphi}\sin^2\theta = F(t) + l_z', \tag{49}$$

where

$$F(t) = \gamma(3r_0^2/2\omega R_0^3)\cos 2\omega t \tag{50}$$

and

[14] Earth's orbit eccentricity is $e_1 \simeq 0.017$ and Moon's orbit eccentricity is $e_2 \simeq 0.055$.

[15] The corrections brought about by this approximation amount to the second decimal, or the first decimal at most, in relevant quantities.

[16] The ratio of these Earth-Moon frequencies is $\Omega/\omega \simeq 1/13$.

[17] This is the usual Poincare-Lindstedt procedure.

$$l_z = mr_0^2\omega'(1-\theta_0^2/2) \tag{51}$$

is a constant of integration. It is reminiscent of the z-component of the unperturbed angular momentum $l_z^{(0)}$, renormalized by γ-interaction (through frequency ω'). Equation (49) expresses the motion of the z-component of the angular momentum in the presence of the perturbation. It leads to equation

$$2m\omega r_1 + mr_0\dot{\varphi}_1 = (3r_0/2\omega R_0^3)\cos 2\omega t \tag{52}$$

for the functions r_1 and $\dot{\varphi}_1$.

Similarly, by making use of (49), equation on the second row in (48) leads to another integral of motion

$$(mr^2\dot{\theta})^2 + \frac{l_z'^2}{\sin^2\theta} = l'^2\,, \tag{53}$$

where

$$l' = mr_0^2\omega' \tag{54}$$

is another constant of integration (reminiscent of the unperturbed angular momentum $l^{(0)}$, renormalized by γ-interaction). Equation (53) has the same form as the one corrresponding to the unperturbed motion, so it gives no equation for r_1 and θ_1, as it can be checked easily.

Finally, by making use of the two integrals of motion given by (49) and (53), the first equation in (48) leads to

$$m\ddot{r}' - l'^2/mr'^3 + \beta/r'^2 = \gamma(r_0/R_0^3) \tag{55}$$

and

$$m\ddot{r}_1 + (3l^2/mr_0^4)r_1 - (2\beta/r_0^3)r_1 = 6(r_0/R_0^3)\cos 2\omega t\,. \tag{56}$$

Equation (55) gives the shifted frequency

$$\omega' = \omega(1-\gamma r_0^3/2\beta R_0^3) = \omega(1-\Omega^2/4\omega^2) \tag{57}$$

and the unperturbed solution r' in (56), with eccentricity e_2' corresponding to another constant of integration E_2' (unperturbed energy). Equations (52) and (56) can now be easily solved. Their solutions read

$$\begin{aligned} r_1 &= -(2r_0^4/\beta R_0^3)\cos 2\omega t\,, \\ \varphi_1 &= -(5r_0^3/4\beta R_0^3)\sin 2\omega t\,. \end{aligned} \tag{58}$$

The solution of the m-body motion within this approximation is now complete. It is given by (46) and by (58), with shifted frequency ω' given by (57).

Within this approximation $\theta_1 = 0$. One can check that the total energy $E_2 + \gamma v = E_2' - \gamma(r_0^2/R_0^3)$ is constant. The corrections to the angular momentum **l**, as given by (49) and (53), are compensated by similar corrections to the angular momentum **L**, such that the total momentum $\mathbf{L}_{tot}$ conserves. The motion is characterized by three basic frequencies: Ω, ω and ω', though the bare frequency ω is not observable. The calculations can be extended to higher-order terms, where combined frequencies appear, as well as additional contributions to the frequency shift. The method can also be applied to other situations of three bodies interacting through gravitational potentials, like, for instance, two bodies gravitating around a third one (Jupiter and Saturn, for instance, where a natural perturbation is just their own interaction, since their mass is much lighter than Sun's mass, and they do not get too close to each other).[18]

THE FOUR MOONS AND FOUR PERIODICITIES

It is well known that the Moon's orbit exhibits four periodicities, beside $T_0 \simeq 365.26$ days of the year corresponding to frequency Ω. There is, first, the sideral Moon $T_1 \simeq 27.32$ days, then the anomalous Moon $T_2 \simeq 27.55$ days, the nodal Moon $T_3 \simeq 27.21$ days and the synodal Moon $T_4 \simeq 29.53$ days. Making use of the numerical data given herein ($m \simeq 7 \times 10^{22}$ Kg, $M \simeq 6 \times 10^{24}$ Kg, $m_0 \simeq 2 \times 10^{30}$ Kg, $r_0 \simeq 384\,000$ Km, $R \simeq 150\,000$ Km) and the gravitational constant $G = 6.7 \times 10^{-11}$ $\mathrm{m^3/Kg \cdot s^2}$, one gets easily $T_0 \simeq 364.78$ days from $\Omega^2 = \alpha/MR_0^3$, and the bare period $\overline{T} \simeq 27.28$ days, corresponding to the bare frequency $\omega^2 = \beta/mr_0^3$. The sideral Moon corresponds to frequency ω' given by (57), and one can check easily that it implies a frequency shift $\delta\omega/\omega = -\Omega^2/4\omega^2 \simeq -1.4 \times 10^{-3}$. It corresponds to a difference of $\delta T \simeq 0.04$ days, which gives the sideral Moon $T_1 = \overline{T} + \delta T \simeq 27.32$ days. In the rotating frame of the Earth the periodicity is $\omega' - \Omega$, which corresponds to a change $\delta\omega/\omega' = -\Omega/\omega \simeq 0.08$ in frequency.[19] It implies a change $\delta T \simeq 2.2$ days, corresponding to the synodal Moon $T_4 = T_1 + \delta T \simeq 29.52$ days. The nodal Moon is associated with the periodiciy

[18] An analytical series expansion in terms of known functions for the coordinates of the Planets was suggested by Weierstrass as a problem in the contest held around 1890 in honor of Sweden's King. Poincaré won the contest without solving the problem, though he pointed out possible instabilities. A formal series expansion was given later by Sundman.

[19] In general, in the rotating frame the coordinate $\mathbf{r}$ is the same, while the velocity is obtained by using $\dot{\mathbf{r}} - \Omega \times \mathbf{r}$ for time derivative.

of the $\tilde{z}$ coordinate in the rotating frame. It is easy to see, by using directly the transcription of the hamiltonian given by (38) in the rotating frame, that this frequency is given by $\tilde{\omega}^2 = \omega^2 + \Omega^2 = \omega'^2(1+\Omega^2/2\omega^2)+\Omega^2$, which implies a change $\delta\omega/\omega' = 3\Omega^2/4\omega^2$. It corresponds to $\delta T \simeq -0.11$ days, which gives the nodal Moon $T_3 = T_1 - \delta T \simeq 27.21$ days. This correction gives also $(4\omega/3\Omega)T_0 \simeq 18$ years for the slow motion of the Moon's nodal plane.[20] According to (46) and (58) the angle φ reads $\varphi \simeq \omega' t - (5\Omega^2/4\omega)t + ...$ in the limit of short times, which amounts to a change $\delta\omega/\omega' = -3\Omega^2/2\omega^2$ in frequency. It leads to $\delta T \simeq 0.22$ days, *i.e.*, a difference twice as much as the difference between the nodal Moon and the sideral Moon, which may be associated with Moon's anomaly $T_2 = T_1 + \delta T \simeq 27.54$.

A FEW "CHAOTICAL" CONSIDERATIONS

In general, the three-body problem is not integrable. However, non-analytical behaviour may exist, as, for instance, an infinite phase velocity $\dot{\varphi}$ for a vanishing polar angle θ. This may imply an abrupt change in the trajectory (for instance, instead of rotating very fast around the pole, the trajectory may take suddenly a longitudinal circle). Apart from particular initial conditions, such chaotic behaviour of the three-body problem would require an external perturbation, usually time dependent, a situation sometime referred to as "Moon's problem", where the Earth's coordinates act like time-dependent external fields. Nevertheless, the motion described above, very likely, by four fundamental frequencies (as well as by the corresponding "combined" frequencies and their higher harmonics), may look already very complicated to warrant the adjective "erratic", or "chaotic", though over very small scale of magnitude.[21] It is sometime called Poincaré's weak chaos, in contrast to trajectory instabilities that are termed strong chaos.[22]

Laborious contributions to Newton's γ-correction to the motion of Moon's nodes (nodal Moon) and the derive of Moon's perigee (anomalous Moon) were brought by d'Alembert and Clairaut around 1750, while Delauney set about to compute about 500 perturbation terms around 1840 (published in about 2,000 print pages). Hill (~ 1880) studied Moon's problem in the rotating frame of the Earth, while Poincaré (~ 1890–1900) developed further insights into such a complicated mechanical behaviour of the 3-body problem. Modern computers (employed especially in connection with aselenization plans) brought additional insights. The

[20] Correction $3\Omega^2/4\omega^2$ was known to Newton.

[21] In particular, the corresponding orbits are no more closed (or "periodic").

[22] The latter are similar to well-known parametric resonance, or non-linear resonance, in the theory of the linear oscillators, driven by an external control parameter.

difficulties reside in the slow convergence, resonant terms, a required accuracy rather high (a small error on the Earth may result in a big failure on the Moon!), and computing algorithms. Meanwhile, chaotical behaviour was left to be looked for in the quantal behaviour, where quantization is attempted for erratic classical trajectories.

A PARTICULAR MOTION IN COULOMB POTENTIAL

The energy in Coulomb (or gravitational) potential $-\alpha/r$ reads

$$E = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r\,, \tag{59}$$

where m is the particle mass and L is the angular momentum. Let $L = 0$ for the moment, and $E = -\alpha/r_0$. The particle will pass through the origin up to the second r_0, then will return to the former r_0, in a periodic movement. Formally, it can be viewed as oscillating between $r = 0$ and $r = r_0$, around $r_0/2$. By (59), it is easy to get

$$m\dot{r}^2/2 + \alpha(r - r_0)/rr_0 = 0, \tag{60}$$

or, by $r = r_0/2 + \rho$,

$$m\dot{\rho}^2/2 + \alpha(\rho - r_0/2)/r_0(\rho + r_0/2) = 0. \tag{61}$$

It is also convenient to use $\rho = r_0 u/2$, so (61) becomes

$$\dot{u}^2 + \omega^2(u-1)/(u+1) = 0, \tag{62}$$

where

$$\omega^2 = 8\alpha/mr_0^3\ . \tag{63}$$

It is easy to integrate equation (62). One obtains

$$2\arcsin\sqrt{(1-u)/2} + \sqrt{1-u^2} = \omega t\,, \tag{64}$$

and the solution must be periodically extended to any t. It describes a periodic motion with period $T = 4\pi/\omega$, as if the frequency would be $\omega/2$.

QUANTIZATION

Equation (61) gives also

$$m\dot{\rho}^2/2 + m\omega^2\rho^2/2 + ... - \alpha/r_0 = 0 \tag{65}$$

by expansion in powers of ρ, which describes a linear harmonic oscillator. It follows

$$\hbar\omega(n+1/2)/2 = \alpha/r_0\,, \tag{66}$$

where frequency $\omega/2$ is used (as for the complete motion) and $n = 0, 1, 2, \ldots$. According to the quantal hypothesis, one can also write it as

$$\hbar\omega\delta n/2 = |E|_q, \tag{67}$$

where $\delta n = n = 1, 2, 3\ldots$ and E_q is the quantized energy. Making use of $\omega = (8|E|_q^3/m\alpha^2)^{1/2}$ one gets

$$|E|_q = \frac{m\alpha^2}{2\hbar^2 n^2}, \tag{68}$$

i.e., the quantized energy of the Hydrogen atom. The anharmonic corrections to (65) do not contribute, as it can be seen from the variation equation (67). The corresponding approximate wavefunctions of the linear oscillator must be displaced so as to be peaked on the origin.

Similarly, for $L \neq 0$, the effective potential in (59) has a minimum value for

$$r_0 = L^2/m\alpha, \tag{69}$$

and energy reads

$$E = m\dot{r}^2/2 + (\alpha/2r_0^3)(r-r_0)^2 + \ldots - \alpha/2r_0\,. \tag{70}$$

The frequency is given by $m\omega^2 = \alpha/r_0^3$, and

$$\hbar\omega(n+1/2)/2 = \alpha/2r_0 + E. \tag{71}$$

Since $L^2/2I = L^2/2mr_0^2 = \alpha/2r_0 = |E|_q$, it is easy to see that (71) leads to

$$\sqrt{2\hbar^2|E|_q^3/m\alpha^2}\,\delta n = |E|_q, \tag{72}$$

which is again the quantal energy (68) of the Hydrogen atom, for $\delta n = n = 1, 2, 3\ldots$. The corresponding approximate wavefunctions of linear harmonic oscillator are now peaked on r_0.

The method can be generalized to any central-field potential $v(r)$. The minimum of the effective potential is reached for r_0 given by

$$-L^2/mr_0^3 + v_1 = 0 \tag{73}$$

where v_1 is the first derivative of v for r_0. The energy expansion reads

$$E = m\dot{r}^2/2 + (3v_1/r_0 + v_2)(r - r_0)^2/2 + ... + L^2/2mr_0^2 + v_0 , \qquad (74)$$

where v_0 is the potential function for r_0, v_2 is the second derivative of v for r_0, and frequency is given by

$$\omega^2 = 3v_1/mr_0 + v_2/m . \qquad (75)$$

The quantization relation reads

$$\hbar\omega(n + 1/2)/2 = E - L^2/2mr_0^2 - v_0 . \qquad (76)$$

Making use of (73) the energy can be related to r_0 by $\left|E\right|_q = L^2/2mr_0^2 = v_1 r_0/2.$ Consequently, the quantized energy is given by

$$\hbar^2[3v_1(\left|E\right|_q)/mr_0(\left|E\right|_q) + v_2(\left|E\right|_q)/m]n^2/4 = \left|E\right|_q^2 , \qquad (77)$$

where $n = 1, 2, 3\dots$